

ON EXTENSIONS OF TYPICAL GROUP ACTIONS

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ABSTRACT. For every countable abelian group G we find the set of all its subgroups H ($H \leq G$) such that a typical measure-preserving H -action on a standard atomless probability space (X, \mathcal{F}, μ) can be extended to a free measure-preserving G -action on (X, \mathcal{F}, μ) . The description of all such pairs $H \leq G$ was made in purely group terms, in the language of the dual \widehat{G} , and G -actions with discrete spectrum. As an application, we answer a question when a typical H -action can be extended to a G -action with some dynamic property, or to a G -action at all. In particular, we offer first examples of pairs $H \leq G$ satisfying both G is countable abelian, and a typical H -action is not embeddable in a G -action.

1. INTRODUCTION

By a transformation T we mean an invertible measure-preserving map defined on a non-atomic standard Borel probability space (X, \mathcal{F}, μ) . Iterations of this map that is sometimes called an automorphism define an action of \mathbb{Z} , thus forming a subgroup of the group of all transformations of (X, \mathcal{F}, μ) . The *spectral* properties of T are those of the induced (Koopman's) unitary operator on $L^2(\mu)$ defined by

$$\widehat{T} : L^2(\mu) \rightarrow L^2(\mu); \quad \widehat{T}f(x) = f(Tx).$$

Transformations and, a bit more generally, group actions (i.e. group representations by transformations) are main objects to study in modern ergodic theory. Investigations describing roughly what is changed if we go to actions of larger groups or back are one of the steadily well-developing aspects of ergodic theory. In this paper, we take a global view and, rather than study specific group actions, we are interested in the spaces of all group actions.

Definition 1.1. We say that groups are **weakly isomorphic** if they are isomorphic to some subgroups each other.

It is easy to check that this is an equivalence relation, and two groups G_+ and G_- are weakly isomorphic if and only if there exist two groups $G_\pm^* \geq G_\mp$ which are isomorphic to G_\pm respectively. Let us remind that a group is called *cohopfian* if it is not isomorphic to any proper subgroup, and *bounded* if there exists an upper bound of the orders of its elements. It is easy to see that G is cohopfian if and only

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if it is not weakly isomorphic to any proper subgroup, i.e. the set of its subgroups that are weakly isomorphic to G is trivial. In particular, if G is cohopfian then a weak isomorphism of H and G actually implies an isomorphism of H and G . However, there exist a lot of weakly isomorphic and not isomorphic pairs even in the class of (infinite) bounded countable abelian groups. It clearly follows from an explicit description of weakly isomorphic pairs (see Section 6).

We say that a typical group action has some property (or the property is said to be typical) if the set of elements satisfying the property contains a dense G_δ subset. Following [13], a property B is called **dynamic** if it is invariant with respect to any measure-preserving isomorphism φ (i.e. $B = \varphi^{-1}B\varphi$) and forms a Baire set (i.e. an almost open set $B = U\Delta M$, where U is open and M is meager). By the well-known topological 0 – 1 law (see [13], [14]), which applies to the standard space of group actions Ω_G for any countable group G , every dynamic property is meager or typical.

The complete classification of countable abelian group pairs $H \leq G$ under the condition for a typical H -action to admit an extension to a free G -action comes from the following main theorem.

Theorem 1.2. *Let G be any countable abelian group, H its subgroup. Suppose H is not an infinite bounded group; then a typical H -action can be extended to a free G -action. Suppose H is an infinite bounded group; then a typical H -action can be extended to a free G -action if and only if G is weakly isomorphic to H .*

The choice of a dynamic property ” to be free” was made there because, in particular, any free G -action T can not be reduced to an action of a ”smaller” group, i.e. the map $T : G \rightarrow \Omega$ is an isomorphism. Besides, the set F of all free G -actions is **characteristic** among all dynamic properties for the restriction map $\pi_H : \Omega_G \rightarrow \Omega_H$, where $\pi_H(T) = T|_H$. Namely, a typical H -action can be extended to a free G -action, equivalently, $\pi_H(F)$ is a typical set, if and only if maps $\pi_H^{\pm 1}$ send every second category/typical set onto a second category/typical set only (see Subsect. 6.1). This means that, for a typical H -action, an extension to a free G -action implies an extension to a G -action satisfying any particular typical dynamic property.

The subject we treat in this paper was indirectly initiated by King who proved that a typical transformation admits at least one root of any fixed order (see [24]). It can be viewed as a typical $n\mathbb{Z}$ -action can be extended to a \mathbb{Z} -action. The whole \mathbb{Z} -actions are clearly free if we extend free $n\mathbb{Z}$ -actions. In answer to Rudolph-del Junco’s and King’s questions (see [19], [24]), it was proved by Ageev (see [2]) that a typical transformation is not prime and has infinitely many roots of any order. Actually, it was an easy corollary of the fact that a typical transformation is embeddable in a free $\mathbb{Z} \oplus G$ -action for any finite abelian group G . Let us also mention papers [28], [32], proving the embeddability of a typical transformation in actions of some non-discrete groups.

Recently Melleray proved Theorem 1.2 for H being finitely-generated via category-preserving maps and a generalization of the classical Kuratowski-Ulam theorem (see [26]).

In order to prove Theorem 1.2, we first follow the traditional way based on the locally dense points technique by extending the proof of Theorem 1.2 for $H = \mathbb{Z}$ (see [3]) as much as it is possible. The novelty is we construct a different explicit set of locally dense points for π_H (see Sections 3, 5). So we get one more proof of

the main results of [24], [2], and [26]. Finally, to check no free extendability for group pairs $H \leq G$ left, we apply a group version of the well-known weak-closure theorem that we prove for every infinite countable abelian group.

As a bonus, we conclude (see Theorem 6.5) that it is closely related to extensions of ergodic H -actions with discrete spectrum. As an application of the main theorem we then provide the complete description of all group pairs $H \leq G$, where G is countable abelian, admitting some extension to a G -action for a typical H -action (see Theorems 6.7-9). Moreover, we show that if a typical H -action can be extended to a G -action then, in fact, it is because π_H restricted to natural subspaces in Ω_G send every second category/typical set to a second category/typical set (see Theorem 6.7, Remark 6.11). As another application of the main theorem we describe all countable abelian groups G with generically monothetic centralizer for a typical G -action (see Theorem 6.12).

The paper is organized as follows. In Sect. 2 we collect all needed technical facts. In Sect. 3 we offer a written version of the proof of Theorem 2 announced in [3] (i.e. Theorem 1.2 for $H = \mathbb{Z}$ only) as a model case. Sect. 4 contains two key subtheorems on locally dense points for π_H and on the centralizer of G -actions that are of the independent interest for possible applications, because, in particular, of its large constructive potential. Theorem 1.2 and its equivalent versions are proved in Sect. 6 as a consequence of results from Sect. 3-5. In Sect. 6 we also show what is changed in the description of pairs $H \leq G$ if we wish to extend a typical H -action to a G -action satisfying any particular typical dynamic property. Finally, in Sect. 7 we discuss a non-abelian case and related questions.

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2. METRICS, FINITENESS, APPROXIMATIONS, LOCALLY DENSE POINTS

Let us identify measure-preserving transformations of (X, \mathcal{F}, μ) which are equal for almost every x of X . On the set Ω of all such transformations the weak (coarse) topology is defined by the convergency $T_n \rightarrow T$ iff $\mu(T_n^{-1}A\Delta T^{-1}A) \rightarrow 0$ for any measurable A . It is equivalent to $\mu(T_n A \Delta T A) \rightarrow 0$ for any measurable A . Let $\xi_n \rightarrow \varepsilon$ for some sequence of finite measurable partitions ξ_n , and σ_n be the finite σ -algebra of all the ξ_n -measurable sets. Consider the metric d on Ω , where

$$d(T, S) = \sum \frac{1}{n^2} d_n(T, S), \quad d_n(T, S) = \max_{A \in \sigma_n} \mu(TA \Delta SA).$$

This is a left-invariant metric defining the same topology. The d -metric is not complete, but Ω becomes a Polish topological group with respect to the metric $d(T, S) + d(T^{-1}, S^{-1})$. It is easy to see that if $\langle T_1, \dots, T_n, S_1, \dots, S_n \rangle$ is an abelian group then

$$d(T_1 \cdots T_n, S_1 \cdots S_n) \leq \sum_i d(T_i, S_i). \quad (1)$$

The set Ω_G of all the G -actions becomes a Polish space when it is equipped with the weak topology coming from the convergency

$$T(n) \rightarrow T \text{ as } n \rightarrow \infty \Leftrightarrow (\forall g \in G) T_g(n) \rightarrow T_g \text{ as } n \rightarrow \infty.$$

2.1. Finiteness.

Definition 2.1. Let H be some subgroup of G . We say that a G -action T is **H-finite** if $\langle T_g : g \in H \rangle$ is a finite subgroup in Ω . We say that a G -action is **finite** if it is G -finite.

We need the following simple lemma.

Lemma 2.2. *Every finite H -action P can be extended to a (H -finite) G -action for any countable abelian group G ($H \leq G$).*

Proof. The partition of X into P -orbits is measurable. We say that orbits of points x and y are equal if for any $h \in H$ $P_h x = x$ if and only if $P_h y = y$. It defines an equivalence relation on X partitioning X into finitely many P -invariant measurable sets of points with equal orbits. It is well enough to extend the P -action on each set separately. Let X consist of equal to each other orbits only. Then there exists a measurable P -invariant partition $\xi = \{B_1, \dots, B_n\}$ of X such that P acts on X/ξ transitively and ξ separates different elements of each orbit, i.e. $B_i \ni P_g x \neq x \in B_j$ implies $i \neq j$.

Take any $g_1 \in G \setminus H$. If for all $k \neq 0$ $kg_1 \notin H$, then let P_{g_1} be the identity map. If not, then put $k = \min l > 0 : lg_1 \in H$. We can split every B_j into k $B_j(i)$ measurable sets of equal measure such that $P_h B_j(i) = B_{m(j,h)}(i)$ for any $h \in H, j, i$. It suffices to define P_{g_1} on, say B_1 . Put $P_{g_1} B_1(i) = B_1(i+1)$ if $1 \leq i < k$, and $P_{g_1} B_1(k) = P_h B_1(1)$, where $h = kg_1$. Obviously, we can define $P_{g_1}|_{B_1(k)}$ such that for any $x \in B_1(1)$ $P_{g_1}^k x = P_h x$. By the commutativity, P_{g_1} can be naturally defined on the whole X . It is clear that P is a well-defined finite $\langle H, g_1 \rangle$ -action.

Iterating the above process we get an H -finite G -action we need. □

2.2. Approximations by finite actions. Let G be any countable abelian group. Then $G = \langle g'_1, \dots, g'_k, \dots \rangle$. Put $G_k = \langle g'_1, \dots, g'_k \rangle$. Consider a sequence of positive integers q_n , where $q_n | q_{n+1}, n = 1, \dots$, and for every positive integer k there exists n such that $k | q_n$. By ξ_n denote the partition of $X = [0, 1)$ into q_n half-open intervals of equal length.

Let $L_{n,k}$ be the set of all G_k -actions P preserving the partition ξ_n such that for any g, j $P_g|_{C_j(n)}$ is just a **shift** Q , i.e. $Qx = x + \alpha, x \in [0, 1)$, where α depends on g, j and $\xi_n = \{C_1(n), \dots, C_{q_n}(n)\}$. Obviously, P_g is a permutation of X/ξ_n for any g and P is a finite action. Moreover, $L_{n,k}$ is a finite set, and if $P_g C_j(n) = C_j(n)$, then $P_g|_{C_j(n)}$ is the identity map. By Lemma 2.2 we can extend each P (in many ways) to some G -action P' . Fixing a representative P' to each $P \in L_{n,k}$, we get a finite collection, noted $L_{n,k}^*$, in Ω_G . Here each P belongs to many $L_{n,k}$, but its representatives $P'(n, k)$ might be different to each other.

If G is not a *torsion* group, then put $L'_{n,k} = \{P \in L_{n,k} : P \text{ acts transitively on } X/\xi_n\}$, $n, k \in \mathbb{N}$, and $L_{n,k}^{*'} = \{P' \in L_{n,k}^* : P'|_{G_k} \in L'_{n,k}\}$. If G is an infinite torsion group, then $L'_{n,k}(L_{n,k}^{*'})$ is defined as before for only change the sequence q_n by $q'_n = \#G_n$. If G is a finite group, then $L'_{n,k}$ is not defined.

The aim of this subsection is to prove the following lemma.

Lemma 2.3. *For any positive integers n_0, k_0 , the sets*

$$\bigcup_{n>n_0} \bigcup_{k>k_0} L_{n,k}^* \text{ and } \bigcup_{n>n_0} \bigcup_{k>k_0} L_{n,k}^{l*}$$

are dense in Ω_G .

Proof. It is well known that the set of all free actions is dense in Ω_G .

Fix some neighborhood of a free action T in Ω_G . It contains a cylindric (open) set

$$N(\gamma', \tilde{g}_1, \dots, \tilde{g}_l) = \{S \in \Omega_G : d(S_{\tilde{g}_i}, T_{\tilde{g}_i}) < \gamma', i = 1, \dots, l\}, (\gamma' > 0),$$

where d is a metric on Ω . Choose $k > k_0$ such that both $\tilde{g}_i \in G_k$ $i = 1, \dots, l$, and if G is not a torsion group then G_k contains at least one element of infinite order. Next it is convenient to represent G_k as the direct sum of cyclic subgroups. Namely, let

$$G_k = \bigoplus_{i=1}^r \langle g_i \rangle,$$

where $\deg g_i = p_i, i = 1, \dots, r, p_i = +\infty, i < r_0, p_i$ are primes for $i = r_0, \dots, r$, $\deg g$ is the order of g . It is clear that, for some $\gamma, \gamma'' > 0$ and for a positive integer n'

$$N(T, \gamma, g_1, \dots, g_r, \xi_{n'}) \subseteq N(\gamma'', g_1, \dots, g_r) \subseteq N(\gamma', \tilde{g}_1, \dots, \tilde{g}_l),$$

here

$$N(T, \gamma, g_1, \dots, g_r, \xi) = \{S \in \Omega_G : (\forall D \in \xi) \mu(S_{g_i} D \Delta T_{g_i} D) < \gamma, i = 1, \dots, r\}$$

for any finite measurable partition ξ . Take $m > \max_{i \geq r_0} p_i$, consider sets

$$K_m = \{g \in G_k : g = \sum_i j_i g_i \text{ for some } 0 \leq j_i < p'_i = \min\{p_i, m\}, i = 1, \dots, r\}.$$

By the multidimensional Rokhlin lemma, for any $\varepsilon > 0$ we can find a measurable set, say A , such that $\bigsqcup_{g \in K_m} T_g A$ is X up to ε -measure. Therefore, we get a finite partition η' of X . Let us slightly change each T_{g_i} by T'_{g_i} , $i < r_0$. Namely, put $T'_{g_i} x = T_{g_i}^{1-m} x$ for any $x \in T_g A, g \in \{g \in K_m : j_i(g) = m-1\}$, and let T' be the identity G_k -action on $Y = X \setminus \bigsqcup_{g \in K_m} T_g A$. It is clear that T' is a finite G_k -action, and for any i $T_{g_i}^{p'_i}$ is the identity map. Moreover, for sufficiently small ε and m large enough any extension of T' , noted T'^* , is an element of $N(T, \gamma/2, g_1, \dots, g_r, \xi_{n'})$. Denote $\eta = \eta' \vee \bigvee_{g \in G_k} T'_g \xi_{n'}$. Since η is a finite subpartition of $\xi_{n'}$, there exists $\delta > 0$ such that

$$N(T'^*, \delta, g_1, \dots, g_r, \eta) \subseteq N(T'^*, \gamma/2, g_1, \dots, g_r, \xi_{n'}).$$

Let $\{A_1, \dots, A_s\} = \{D \in \eta : D \subseteq A\}$. Since η is a T' -invariant partition, $\eta = \{Y, T'_g A_i, i = 1, \dots, s, g \in K_m\}$. Then every $\bigsqcup_{g \in K_m} T'_g A_i$ is a T' -invariant set. Since $\xi_n \rightarrow \varepsilon$, for $n > n_0$ large enough we can approximate every $T'_g A_i$ and Y very well by some ξ_n -measurable set $A_i(\xi_n, g)$ (Y_n) such that $\#\{D \in \xi_n : D \subseteq A_i(\xi_n, g)\}$ does not depend on g , $\#K_m/\#\xi_n$, and $\eta_n = \{Y_n, A_i(\xi_n, g), i = 1, \dots, s, g \in K_m\}$ is a partition of X . It is easy to see that we can find $P \in L(n, k)$ such that for any $i, j, g \in K_m$ $P_{g_i} A_j(\xi_n, g) = A_j(\xi_n, g + g_i)$, and $P_{g_i}^{p'_i}$ is the identity map, where if $g = \sum_l j_l g_l$ and $j_i = p'_i - 1$, then $A_j(\xi_n, g + g_i) = A_j(\xi_n, \sum_{l \neq i} j_l g_l)$. This means that any extension P^* of P is an element of $N(T'^*, \delta, g_1, \dots, g_r, \eta)$, and $\bigcup_{n>n_0} \bigcup_{k>k_0} L_{n,k}^*$ is dense in Ω_G .

A priori, P constructed above is not transitive on X/ξ_n . However, it has a "good" orbital structure. Namely, $X \setminus Y_n$ consists of P -orbits $O_i = \bigsqcup_{g \in K_m} P_g C_{s_i}(n)$, $C_j(n) \in \xi_n$. Since $\#K_m | \#\{j : C_j(n) \subseteq Y_n\}$, changing P on Y_n if $Y(Y_n)$ are not empty sets, we get the same orbits O_i , $i = 1, \dots, q'$ on whole X .

If G is not a torsion group, then $p_1 = +\infty$, $p'_1 = m$, and we can change P_{g_1} on $B_n = \bigsqcup_i \bigsqcup_{g \in K_m : j_1(g)=m-1} P_g C_{s_i}(n)$ such that $P\xi_n = \xi_n$, P is a G_k -action by permutations of X/ξ_n , and $X = \bigsqcup_{g \in K'_m} P_g C_{s_1}(n)$, where K'_m is defined as K_m , the only difference is we change p'_1 by $p''_1 = q'p'_1$. Since for m, n large enough the measures of sets $Y, Y_n, \bigsqcup_{g \in K_m : j_1(g)=m-1} P_g A, B_n$ are sufficiently small, we get $P \in L'(n, k)$, and $P^* \in N(T, \gamma, g_1, \dots, g_r, \xi_{n'})$.

If G is an infinite torsion group, then extend P to a G_n -action P' by defining $P'_{g'_{k+1}}, \dots, P'_{g'_n}$ consecutively. Namely, put $t = \min t' > 0 : t'g'_{k+1} \in G_k$. Let $P'_{g'_{k+1}}$ be a shift on every $C_j(n)$, $P'_{g'_{k+1}} \xi_n = \xi_n$, $P'_{g'_{k+1}} C_{s_i}(n) = C_{s_{i+1}}(n)$ if $i \neq 0 \pmod{t}$, and $P'_{g'_{k+1}} C_{s_i}(n) = P_g C_{s_{i-t+1}}(n)$ if $i|t$, where $g = tg'_{k+1} \in G_k$. By the commutativity to every P_g , $g \in G_k$, we can define $P'_{g'_{k+1}}$ uniquely on whole X . Obviously, the G_{k+1} -action T' has $\#G_n/\#G_{k+1}$ orbits on X/ξ_n . Iterating this process, we get $P' \in L'(n, n)$, and $P'^* \in N(T'^*, \delta, g_1, \dots, g_r, \eta)$ (Y is the empty set). Lemma 2.3 is proved. \square

2.3. Locally dense points technique. Suppose X and Y are Polish spaces and $\varphi : X \rightarrow Y$ is a continuous map. The subset C of Y is called an *analytic* set if there exists a Borel subset B of X such that $\varphi(B) = C$. Next we will essentially make use of the fact that every analytic set is almost open (see [25]).

We denote by $LocDen\varphi$ the set of all $x \in X$, called *locally dense* points, such that for any neighborhood $U(x)$ of x , the set $\varphi(U(x))$ is dense in some neighborhood of $\varphi(x)$.

Lemma 2.4. (see [24], [2]) *Let $LocDen\varphi$ be dense in X . Then $\varphi(X)$ is not meager in Y . Moreover, $\varphi(A)$ is not meager in Y for every non-meager A in X .*

3. MODEL CASE

Example 3.1. of a free action T of any countable abelian group G having an element g_1 , of an infinite order such that T_{g_1} is ergodic and has discrete spectrum.

We will only construct such an action, because the proof is obvious. At first fix some representation

$$G = \bigcup_k G_k,$$

where $G_k = \langle g_1, \dots, g_k \rangle$ is a subgroup generated by g_1, \dots, g_k , $k \in \mathbb{N}$, $g_{k+1} \notin G_k$, $g_1 = g$. Next we define a sequence of elements $\tilde{\alpha}^{(k)} = (\tilde{\alpha}_1^{(k)}, \dots)$ from $\bar{X} = \mathbb{R}^{\mathbb{N}}$ as follows.

Denote, $\tilde{\alpha}_i^{(1)} = \beta_i$ $i = 1, 2, \dots$, where (irrational) β_i are rationally independent in the sense $\sum_{i=1}^n r_i \beta_i = 0 \pmod{1}$ gives $r_i = 0$ $i = 1, \dots, n$ for any rational r_1, \dots, r_n and $n \in \mathbb{N}$. Fix some sequence of irrationals γ_k .

If for any n , $ng_{k+1} \notin G_k$, then put $\tilde{\alpha}_{k+1}^{(k+1)} = \gamma_k$ and $\tilde{\alpha}_m^{(k+1)} = 0$ as $m \neq k+1$.

If not, take

$$n_0 = \min_{\substack{ng_{k+1} \in G_k \\ n > 0}} n.$$

Therefore $n_0 g_{k+1} = l_1 g_1 + \dots + l_k g_k$ for some $l_i \in \mathbb{Z}$. Put

$$\begin{aligned} \tilde{\alpha}^{(k+1)} &= \frac{l_1}{n_0} \tilde{\alpha}^{(1)} + \dots + \frac{l_k}{n_0} \tilde{\alpha}^{(k)} + e^{(k)}, \\ e_{k+1}^{(k)} &= \frac{1}{n_0}, \quad e_m^{(k)} = 0 \text{ as } m \neq k+1, \quad e^{(k)} = (e_1^{(k)}, e_2^{(k)}, \dots). \end{aligned}$$

Let then $\alpha^{(k)} = \pi_\infty \tilde{\alpha}^{(k)}$, where π_∞ is a natural projection from \bar{X} onto $X = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \times \dots$. It is enough to define for any

$$g = \sum_{i=1}^k m_i g_i$$

the action T by

$$\begin{aligned} T_g &= T_{g_1}^{m_1} \dots T_{g_k}^{m_k}, \\ T_{g_k} x &= x + \alpha^{(k)} \pmod{1} \end{aligned}$$

on (X, μ) , where μ is the Haar measure on X . Here, of course, T_g does not depend on our choice of the representation g as

$$\sum_{i=1}^k m_i g_i.$$

Lemma 3.2. *The action T constructed in Example 3.1 is a locally dense point for a map $\pi_1 : \Omega_G \rightarrow \Omega$, where $\pi_1(T) = T_{g_1}$.*

Proof. Every ergodic automorphism with discrete spectrum has rank 1 (see[17]), i.e. for T_{g_1} we can choose([6]) a monotonic sequence of partitions $\xi'_q = \{C_1(q), \dots, C_{h(q)}(q), D(q)\}$ such that $T_{g_1} C_i(q) = C_{i+1}(q)$ for $i < h(q)$, $\mu(D(q)) \rightarrow 0$ as $q \rightarrow \infty$, and $\xi'_q \rightarrow \varepsilon$. For partitions $\xi_q = \{C_1(q), \dots, C_{h(q)}(q)\}$ we have $\xi_q \rightarrow \varepsilon$. It is more convenient then to consider in Ω the d metric associated to $\{\xi_q\}_{q=1}^\infty$. Pick some neighborhood of T . It contains a cylindric set $N(\gamma', g'_1, \dots, g'_l) = \{S \in \Omega_G : d(S_{g'_i}, T_{g'_i}) < \gamma' \ i = 1, \dots, l\}$. Choose k such that $g'_i \in G_k \ i = 1, \dots, l$. We can represent $G_k = \langle g_1, \dots, g_k \rangle$ as the direct sum of the form

$$G_k = \bigoplus_{i=1}^r \langle \tilde{g}_i \rangle,$$

where $\tilde{g}_1, \dots, \tilde{g}_{r_0} (1 \leq r_0 \leq r)$ have infinite order, $p_i (p_i < \infty)$ is an order of \tilde{g}_i for $r_0 < i \leq r$. Moreover,

$$g_1 = k_0 \tilde{g}_1 + l_{r_0+1} \tilde{g}_{r_0+1} + \dots + l_r \tilde{g}_r$$

for some $k_0, l_i \in \mathbb{N}$. It is clear that, for some $\gamma > 0$, if $d(S_{\tilde{g}_i}, T_{\tilde{g}_i}) < \gamma \ i = 1, \dots, r$, then $S \in N(\gamma', g'_1, \dots, g'_l)$. Thus it is good enough to find a $\delta > 0$ and a dense set of \tilde{T}_{g_1} in $U_\delta(T_{g_1})$ such that for some $\tilde{T}_{\tilde{g}_1}, \dots, \tilde{T}_{\tilde{g}_r}, \tilde{T}_{g_{k+1}} \dots$ we have both

$$(\tilde{T}_{\tilde{g}_1}, \dots, \tilde{T}_{\tilde{g}_r}, \tilde{T}_{g_{k+1}} \dots) \text{ forms a } G - \text{action}, \quad (2)$$

and

$$d(\tilde{T}_{\tilde{g}_i}, T_{\tilde{g}_i}) < \gamma \ i = 1, \dots, r, \quad (3)$$

where $U_\delta(R) = \{S \in \Omega : d(R, S) < \delta\}$.

Because T_{g_1} has rank 1, we have $C(T_{g_1}) = \overline{\{T_{g_1}^m\}_{-\infty}^{+\infty}}$ in Ω . Denote $t = p_{r_0+1} \cdots p_r$. Therefore take $m_1, \dots, m_r \in \mathbb{N}$ such that

$$d(T_{g_1}^{m_i}, T_{\tilde{g}_i}) < \frac{\gamma}{6} \quad i = 1, \dots, r, \quad (4)$$

$$d(T_{g_1}^{m_i p_i}, E) = d(T_{g_1}^{m_i p_i}, T_{\tilde{g}_i}^{p_i}) < \frac{\gamma}{9t} \quad r_0 < i < r, \quad (5)$$

$$\begin{aligned} d(T_{g_1}^{m_1 k_0} T_{g_1}^{l_{r_0+1} m_{r_0+1}} \cdots T_{g_1}^{l_r m_r}, T_{g_1}) = \\ d(T_{g_1}^{m_1 k_0} T_{g_1}^{l_{r_0+1} m_{r_0+1}} \cdots T_{g_1}^{l_r m_r}, T_{g_1}^{k_0} T_{\tilde{g}_{r_0+1}}^{l_{r_0+1}} \cdots T_{\tilde{g}_r}^{l_r}) < \frac{\gamma}{9}. \end{aligned} \quad (6)$$

Let $L(q)$ be the set of cyclic permutations ξ_q , i.e. $L(q) = \{T_u \in \Omega : T_u \xi_q = \xi_q \text{ \& } T_u^{h(q)} = E \text{ \& } \forall i (T_u^m C_i(q) = C_i(q) \Leftrightarrow m \equiv 0 \pmod{h(q)})\}$. Since $\xi_q \rightarrow \varepsilon$, it implies (see [15]) the density of $\cup_{q>q_0} L(q)$ in Ω for any q_0 . Take q_0 such that

$$\sum_{q>q_0}^{\infty} \frac{1}{q^2} d_q(T, S) < \frac{\gamma}{18t} \quad (7)$$

for any $T, S \in \Omega$. Using (6), choose $\delta > 0$ such that

$$\forall T_k \in \Omega : d(T_{g_1}, T_k) < \delta \Rightarrow d(T_{g_1}^{m_i}, T_k^{m_i}) < \frac{\gamma}{6} \text{ \& } \quad (8)$$

$$d(T_{g_1}^{m_i p_i}, T_k^{m_i p_i}) < \frac{\gamma}{9t}, \quad r_0 < i \leq r \text{ \& } \quad (9)$$

$$d(T_k^{m_1 k_0} T_k^{l_{r_0+1} m_{r_0+1}} \cdots T_k^{l_r m_r}, T_k) < 2\frac{\gamma}{9}. \quad (10)$$

For every

$$T_u \in \bigcup_{q>q_0} L(q) \cap U_{\delta}(T_{g_1}),$$

let us define $\tilde{T}_{\tilde{g}_1}, \dots, \tilde{T}_{\tilde{g}_r}$ such that (2) and (3) hold. (Here $T_u = \tilde{T}_{g_1}$, and the choice of remaining $\tilde{T}_{g_{k+1}}, \tilde{T}_{g_{k+2}}, \dots$ is more or less obvious, because all G_k -actions below are finite).

Consider ξ_q , where q is defined by $T_u \in L(q)$ ($q > q_0$). Change the numbering of $C_j(q)$ by $C_{(i)}(q)$, where i is considered modulo $h(q)$ and is found from $T_u^i C_1(q) = C_j(q)$. Then

$$T_u C_{(i)} = C_{(i+1)}(q) \quad (i \pmod{h(q)}).$$

Cut every $C_{(i)}(q)$ into $k_0 \cdot p_{r_0+1} \cdots p_r$ measurable sets $C_{(i)}(i_{r_0}, \dots, i_r)$ of the equal measure, where the collection (i_{r_0}, \dots, i_r) is considered modulo $(k_0, p_{r_0+1}, \dots, p_r)$ and

$$T_u C_{(i_0)}(i_{r_0}, \dots, i_r) = C_{(i_0+1)}(i_{r_0}, \dots, i_r)$$

for any $(i_0, i_{r_0}, \dots, i_r) \pmod{h(q)}$.

It is clear that we can find $\tilde{T}'_{\tilde{g}_j}$ $j = r_0, r_0 + 1, \dots, r$ such that

$$\tilde{T}'_{\tilde{g}_j} C_{(i_0)}(i_{r_0}, \dots, i_r) = C_{(i_0)}(i_{r_0}, \dots, i_{j-1}, i_j + 1, i_{j+1}, \dots, i_r),$$

$T_u, \tilde{T}'_{\tilde{g}_{r_0}}, \dots, \tilde{T}'_{\tilde{g}_r}$ form an abelian group, and have order $h(q), k_0, p_{r_0+1}, \dots, p_r$ respectively. Moreover, this group is just the direct sum of cyclic subgroups generated by them.

From our choice above $\tilde{T}_{\tilde{g}_j}$ are needed to be very close to $T_u^{m_j}$. For $x \in C_{(i)}(i_{r_0}, \dots, i_r)$ let

$$\tilde{T}_{\tilde{g}_j} = T_u^{m_j} \text{ for } 1 < j \leq r_0,$$

$$\tilde{T}_{\tilde{g}_j}x = \begin{cases} T_u^{m_j}\tilde{T}'_{\tilde{g}_j}x & \text{if } i_j \neq p_j - 1 \\ T_u^{m_j-p_j m_j}\tilde{T}'_{\tilde{g}_j}x & \text{if } i_j = p_j - 1 \end{cases} \text{ for } r_0 < j \leq r,$$

$$\tilde{T}_{\tilde{g}_1}x = \begin{cases} T_u^{m_1}\tilde{T}'_{\tilde{g}_{r_0}}x & \text{if } i_{r_0} \neq k_0 - 1 \\ \tilde{T}_{\tilde{g}_{r_0+1}}^{-l_{r_0+1}} \dots \tilde{T}_{\tilde{g}_r}^{-l_r} T_u^{1+m_1-k_0 m_1} \tilde{T}'_{\tilde{g}_{r_0}}x & \text{if } i_{r_0} = k_0 - 1. \end{cases}$$

It is clear that automorphisms (or permutations of $C_{(i)}(i_{r_0}, \dots, i_r)$) $\tilde{T}_{\tilde{g}_1}, \tilde{T}_{\tilde{g}_2}, \dots, \tilde{T}_{\tilde{g}_r}$ form a non-free G_k -action such that $T_u = \tilde{T}_{\tilde{g}_1}^{k_0} \tilde{T}_{\tilde{g}_{r_0+1}}^{l_{r_0+1}} \dots \tilde{T}_{\tilde{g}_r}^{l_r}$. For $1 < j \leq r_0$, using (4),(8), we have

$$d(\tilde{T}_{\tilde{g}_j}, T_{\tilde{g}_j}) = d(T_u^{m_j}, T_{\tilde{g}_j}) \leq d(T_u^{m_j}, T_{g_1}^{m_j}) + d(T_{g_1}^{m_j}, T_{\tilde{g}_j}) < \frac{\gamma}{3}.$$

In order to confirm (3) for $r_0 < j \leq r$, we take an element $A \in \xi_{q_1}$, where $q_1 \leq q_0$. The set A consists of a union of some $C_{(i)}(q)$. Let

$$H_0(j) = \bigcup_{i, i_{r_0}, \dots, i_r} C_{(i)}(i_{r_0}, \dots, i_{j-1}, 0, i_{j+1}, \dots, i_r).$$

By definition,

$$\begin{aligned} \mu(T_u^{-m_j} A \Delta \tilde{T}_{\tilde{g}_j}^{-1} A) &= \mu(T_u^{m_j} A \Delta \tilde{T}_{\tilde{g}_j} A) = \\ 2\mu(T_u^{m_j} A \setminus \tilde{T}_{\tilde{g}_j} A) &= 2\mu((T_u^{m_j} A \cap H_0(j)) \setminus (\tilde{T}_{\tilde{g}_j} A \cap H_0(j))) = \\ 2\mu((T_u^{m_j} A \cap H_0(j)) \setminus (T_u^{m_j-p_j m_j} A \cap H_0(j))) &= \frac{2}{p_j} \mu(A \setminus T_u^{-p_j m_j} A) = \\ \frac{1}{p_j} \mu(T_u^{p_j m_j} A \Delta A). \end{aligned}$$

Therefore,

$$d_{q_1}(T_u^{m_j}, \tilde{T}_{\tilde{g}_j}) = \frac{1}{p_j} d_{q_1}(T_u^{p_j m_j}, E) \text{ for } q_1 \leq q_0.$$

Now if we recall (5), (7), (9), we get

$$d(T_u^{m_j}, \tilde{T}_{\tilde{g}_j}) \leq \frac{1}{p_j} d(T_u^{p_j m_j}, E) + \frac{\gamma}{9t} \leq \frac{\gamma}{3t}, \quad (11)$$

Combining this with (4), (8), we obtain

$$d(\tilde{T}_{\tilde{g}_j}, T_{\tilde{g}_j}) \leq d(\tilde{T}_{\tilde{g}_j}, T_u^{m_j}) + d(T_u^{m_j}, T_{g_1}^{m_j}) + d(T_{g_1}^{m_j}, T_{\tilde{g}_j}) \leq 2\frac{\gamma}{3}. \quad (12)$$

For remaining $j = 1$, as above, we have

$$\begin{aligned} \mu(T_u^{m_1} A \Delta \tilde{T}_{\tilde{g}_1} A) &= \frac{2}{k_0} \mu(A \setminus \tilde{T}_{\tilde{g}_{r_0+1}}^{-l_{r_0+1}} \dots \tilde{T}_{\tilde{g}_r}^{-l_r} T_u^{1-k_0 m_1} A) = \\ \frac{1}{k_0} \mu(T_u A \Delta T_u^{k_0 m_1} \tilde{T}_{\tilde{g}_{r_0+1}}^{l_{r_0+1}} \dots \tilde{T}_{\tilde{g}_r}^{l_r} A); & \\ d(T_u^{m_1}, \tilde{T}_{\tilde{g}_1}) &\leq \frac{1}{k_0} d(T_u, T_u^{k_0 m_1} \tilde{T}_{\tilde{g}_{r_0+1}}^{l_{r_0+1}} \dots \tilde{T}_{\tilde{g}_r}^{l_r}) + \frac{\gamma}{9t}. \end{aligned} \quad (13)$$

Taking into account (1), from (11) we have

$$d(T_u^{m_j s}, \tilde{T}_{\tilde{g}_j}^s) \leq \frac{s\gamma}{3t}$$

for $j > r_0$, $s \in \mathbb{N}$. Thus

$$d(\tilde{T}_{\tilde{g}_{r_0+1}}^{l_{r_0+1}} \dots \tilde{T}_{\tilde{g}_r}^{l_r}, T_u^{l_{r_0+1} m_{r_0+1}} \dots T_u^{l_r m_r}) \leq \frac{\gamma}{3}.$$

Combining this with (10),(13), we obtain

$$d(T_u^{m_1}, \tilde{T}_{\tilde{g}_1}) \leq \frac{1}{k_0} d(T_u, T_u^{k_0 m_1} T_u^{l_{r_0+1} m_{r_0+1}} \dots T_u^{l_r m_r}) + \frac{\gamma}{9t} + \frac{\gamma}{3} \leq \frac{2\gamma}{3}.$$

As in (12) we have

$$d(\tilde{T}_{\tilde{g}_1}, T_{\tilde{g}_1}) < \gamma.$$

Lemma 3.2 is proved. \square

3.1. Extensions of typical automorphisms to group actions.

Theorem 3.3. *Let G be any countable abelian group having an element g_1 of infinity order. Then a typical automorphism is embeddable as T_{g_1} in a free group action T of G .*

Proof. Fix G and the natural projection $\pi_1 : \Omega_G \rightarrow \Omega$, where $\pi_1(T) = T_{g_1}$. All free G -actions form a dense G_δ set F in Ω_G . Therefore the set $B = \pi_1(F)$ is almost open as an analytic set. Moreover, it is clearly dynamic. By the topological 0 – 1 law, B is then meager or typical.

Besides, It is well know that conjugates to any free action are dense in Ω_G for any countable abelian group G . Since $LocDen\pi_1$ is invariant with respect to any conjugate, by Lemma 3.2 $LocDen\pi_1$ is dense in Ω_G . From Lemma 2.4 B is then typical and Theorem 3.3 follows. \square

4. TWO KEY SUBTHEOREMS

Theorem 4.1. *Let T be an action of a countable abelian group G and H be a subgroup of G . If $CL\{T_g : g \in G\} = CL\{T_h : h \in H\}$, then T is a locally dense point for the restriction map $\pi_H : \Omega_G \rightarrow \Omega_H$, where $\pi_H(T) = T|_H$.*

Proof. Consider a sequence of positive integers q_n , where $q_n | q_{n+1}$, $n = 1, \dots$, and for every positive integer k there exists n such that $k | q_n$. Let ξ_n be the partition of $X = [0, 1)$ into q_n half-open intervals of equal length. It is clear that the sequence ξ_n is monotonic and $\xi_n \rightarrow \varepsilon$ as $n \rightarrow \infty$. From now on d is the metric in Ω associated to the sequence ξ_n .

Fix some neighborhood of T in Ω_G . It contains a cylindric (open) set

$$N(\gamma', g'_1, \dots, g'_l) = \{S \in \Omega_G : d(S_{g'_i}, T_{g'_i}) < \gamma', i = 1, \dots, l\}, (\gamma' > 0).$$

Choose k_0 such that $g'_i \in G_{k_0}$ $i = 1, \dots, l$. Next it is convenient to represent $G_{k_0} = \langle g_1, \dots, g_{k_0} \rangle$ as the direct sum of the form

$$G_{k_0} = \bigoplus_{i=1}^r \langle \tilde{g}_i \rangle,$$

where $\deg \tilde{g}_i = p_i$, $i = 1, \dots, r$, p_i are primes or $+\infty$. It is clear that, for some $\gamma > 0$

$$N(\gamma, \tilde{g}_1, \dots, \tilde{g}_r) \subseteq N(\gamma', g'_1, \dots, g'_l).$$

All we need to prove is $\pi_H(N(\gamma, \tilde{g}_1, \dots, \tilde{g}_r))$ is dense in some neighborhood $U \subseteq \Omega_H$ of $T|_H$. Let us choose then such U , a dense subset of H -actions \tilde{T} in U , admitting extensions to some G -actions \tilde{T} from $N(\gamma, \tilde{g}_1, \dots, \tilde{g}_r)$. Denote $H_n = G_n \cap H$,

$$s_i = \min_{\substack{m \tilde{g}_i \in \langle H_n, \tilde{g}_1, \dots, \tilde{g}_{i-1} \rangle \\ m > 0}} m,$$

or $s_i = +\infty$ if for any $m > 0$ $m \tilde{g}_i \notin \langle H_n, \tilde{g}_1, \dots, \tilde{g}_{i-1} \rangle$, $i = 1, \dots, r$. If $s_i \neq +\infty$, then

$$s_i \tilde{g}_i = h(i) + \sum_{j < i} k_j(i) \tilde{g}_j,$$

for some $h(i) \in H_n$. It is clear, that, in general, $s_i, h(i), k_j(i)$ depend on n , but, in fact, s_i are monotonic for any i , therefore they are independent of n for sufficiently large $n > N_0$, and then we can fix corresponding $h(i), k_j(i)$ for $s_i \neq +\infty$. Denote $c = 1$ if for any i $s_i = +\infty$. If not, then $c = \max_{i,j} |k_j(i)|$. Put $\lambda = 36(3c)^r$, and $I = \{1 \leq i \leq r : s_i \neq +\infty\}$. There exist $h_1, \dots, h_r \in H$ such that

$$d(T_{h_i}, T_{\tilde{g}_i}) < \frac{\gamma}{\lambda(1+rc)}, \quad i = 1, \dots, r. \quad (14)$$

We can choose $k' > \max\{k_0, N_0\}$ such that $h_i \in H_{k'} \leq H_{k'+1} \leq \dots$, $i = 1, \dots, r$. Besides we can choose a positive integer n_0 such that

$$\sum_{n>n_0}^{\infty} \frac{1}{n^2} d_n(T, S) < \frac{\gamma}{\lambda} \quad (15)$$

for any $T, S \in \Omega$.

Let U be the (cylindric) set of all H -actions S with the following properties

$$d(T_{h_i}, S_{h_i}) < \frac{\gamma}{\lambda}, \quad i = 1, \dots, r, \quad (16)$$

$$d(T_{h_i}^{s_i}, S_{h_i}^{s_i}) < \frac{\gamma}{\lambda}, \quad \text{if } i \in I, \quad (17)$$

$$d(T_{h(i)} T_{h_1}^{k_1(i)} \dots T_{h_{i-1}}^{k_{i-1}(i)}, S_{h(i)} S_{h_1}^{k_1(i)} \dots S_{h_{i-1}}^{k_{i-1}(i)}) < \frac{\gamma}{\lambda}, \quad \text{if } i \in I. \quad (18)$$

By $L_{n,k}$ denote the set of all H_k -actions \tilde{T} preserving the partition $\xi_n = \{C_1(n), \dots, C_{q_n}(n)\}$ such that for any h, j $\tilde{T}_h|_{C_j(n)}$ is a shift. Fixing an extension \tilde{T}' to each $\tilde{T} \in L_{n,k}$, we get a finite subset $L_{n,k}^*$ of Ω_H . By Lemma 2.3,

$$\bigcup_{n>n_0} \bigcup_{k>k'} L_{n,k}^* \cap U \text{ is dense in } U.$$

Let us modify each $\tilde{T}' \in L_{n,k}^*$, ($n > n_0, k > k'$) to a G -action \tilde{T} in the way $\tilde{T}|_{H_k} = \tilde{T}'|_{H_k}$.¹ Then

$$\tilde{T}|_H \in U \Leftrightarrow \tilde{T}' \in U,$$

and applying Lemma 2.3 again, we get

$$\bigcup_{n>n_0} \bigcup_{k>k'} \bigcup_{\tilde{T}' \in L_{n,k}^*} \tilde{T}|_H \cap U \text{ is dense in } U.$$

If, additionally, because of the construction of $\tilde{T}, \tilde{T}' \in L_{n,k}^* \cap U$ ($n > n_0, k > k'$) implies that the corresponding $\tilde{T} \in N(\gamma, \tilde{g}_1, \dots, \tilde{g}_r)$, then Theorem 4.1 is proved.

Fix $\tilde{T}' \in L_{n,k}^*$ ($n > n_0, k > k'$), put $\tilde{T}|_{H_k} = \tilde{T}'|_{H_k}$, i.e. at this step \tilde{T} is an element of $L_{n,k}$. Extend then the H_k -action \tilde{T} to $\langle G_{k_0}, H_k \rangle$ -action \tilde{T} by transformations $\tilde{T}_{\tilde{g}_1}, \dots, \tilde{T}_{\tilde{g}_r}$ defined consecutively as follows.

Denote $\tilde{s}_i = s_i$ if $i \in I$, $\tilde{s}_i = 1$ if $s_i = +\infty$. Cut every $C_j(n)$ into $\tilde{s}_1 \dots \tilde{s}_r$ half-open intervals $C_j(j_1, \dots, j_r)$ of the equal length, where the collection (j_1, \dots, j_r) is considered modulo $(\tilde{s}_1, \dots, \tilde{s}_r)$ and

$$QC_m(j_1, \dots, j_r) = C_t(j_1, \dots, j_r)$$

for any (j_1, \dots, j_r) , where Q is the shift defined by $Q(C_m(n)) = C_t(n)$, $m, t = 1, \dots, q_n$. Therefore every \tilde{T}_h , $h \in H_k$, sends $C_m(j_1, \dots, j_r)$ to some $C_{m'}(j_1, \dots, j_r)$,

¹ Not necessarily $\tilde{T}_h = \tilde{T}'_h$ if $h \in H \setminus H_k$.

and m' is independent of coordinates (j_1, \dots, j_r) . It is clear that if $1 < s_i \neq +\infty$, then we can find transformations T'_{Δ_i} acting as shifts on each $C_j(j_1, \dots, j_r)$ such that

$$T'_{\Delta_i} C_j(j_1, \dots, j_r) = C_j(j_1, \dots, j_{i-1}, j_i + 1, j_{i+1}, \dots, j_r).$$

Obviously, T'_{Δ_i} commute to each other and to any \tilde{T}_h , $h \in H_k$.

Let us finally define $\tilde{T}_{\tilde{g}_i}$ keeping in mind that $\tilde{T}_{\tilde{g}_1}, \dots, \tilde{T}_{\tilde{g}_{i-1}}$ are well defined. Put

$$\begin{aligned} \tilde{T}_{\tilde{g}_i} &= \tilde{T}_{h_i} \text{ if } s_i = +\infty, \\ \tilde{T}_{\tilde{g}_i} &= \tilde{T}_{h(i)} \tilde{T}_{\tilde{g}_1}^{k_1(i)} \dots \tilde{T}_{\tilde{g}_{i-1}}^{k_{i-1}(i)} \text{ if } s_i = 1. \end{aligned}$$

If $1 < s_i \neq +\infty$, then for $x \in C_j(j_1, \dots, j_r)$ let

$$\tilde{T}_{\tilde{g}_i} x = \begin{cases} \tilde{T}_{h_i} T'_{\Delta_i} x & \text{if } j_i \neq s_i - 1 \\ \tilde{T}_{h(i)} \tilde{T}_{\tilde{g}_1}^{k_1(i)} \dots \tilde{T}_{\tilde{g}_{i-1}}^{k_{i-1}(i)} \tilde{T}_{h_i}^{1-s_i} T'_{\Delta_i} x & \text{if } j_i = s_i - 1. \end{cases}$$

To see why $\tilde{T}_{\tilde{g}_i}$ is a well-defined transformation for $1 < s_i \neq +\infty$, it is convenient to look at it as a permutation of the sets $C_j(j_1, \dots, j_r)$. Indeed, it is a composition of T'_{Δ_i} and a map which fixes each element of the partition $\{B_m(i), m = 0, \dots, s_i - 1\}$, where

$$B_m(i) = \bigcup_{j, j_1, \dots, j_r} C_j(j_1, \dots, j_{i-1}, m, j_{i+1}, \dots, j_r).$$

Moreover, restricted to some level $B_m(i)$, this map is an element of $< \tilde{T}|_{H_k}, \tilde{T}_{\tilde{g}_1}, \dots, \tilde{T}_{\tilde{g}_{i-1}} >$, so it is a permutation of the sets $C_j(j_1, \dots, j_r)$ that makes no change in coordinates j_i, \dots, j_r . Thus $\tilde{T}_{\tilde{g}_i}$ does not change coordinates j_{i+1}, \dots, j_r , and does not depend on them.

Along the same line, it is easy to check, that $\tilde{T}_{\tilde{g}_i}$ commutes to each element of $< \tilde{T}|_{H_k}, \tilde{T}_{\tilde{g}_1}, \dots, \tilde{T}_{\tilde{g}_{i-1}} >$ provided the later is an abelian group.

Assuming \tilde{T} is a well-defined $< H_k, \tilde{g}_1, \dots, \tilde{g}_{i-1} >$ -action, we get that \tilde{T} is well defined as an $< H_k, \tilde{g}_1, \dots, \tilde{g}_i >$ -action too. Indeed, any relation $m\tilde{g}_i = h_0 \in < H_k, \tilde{g}_1, \dots, \tilde{g}_{i-1} >$ ($m \neq 0$) implies $i \in I$, $s_i | m$, and $\tilde{T}_{\tilde{g}_i}^m = \tilde{T}_{h_0}$, because of $\tilde{T}_{\tilde{g}_i}^{s_i} = \tilde{T}_{h(i)} \tilde{T}_{\tilde{g}_1}^{k_1(i)} \dots \tilde{T}_{\tilde{g}_{i-1}}^{k_{i-1}(i)}$. This means that we extended above \tilde{T} to a finite $< G_{k_0}, H_k >$ -action. By Lemma 2.2, we can extend the later to some G -action \tilde{T} .

In order to show that $\tilde{T} \in N(\gamma, \tilde{g}_1, \dots, \tilde{g}_r)$ if $\tilde{T}' \in U$, let us prove by the induction that

$$d(\tilde{T}_{\tilde{g}_i}, \tilde{T}_{h_i}) < \frac{\gamma}{3(3c)^{r-i}}, \quad i = 1, \dots, r. \quad (19)$$

Note that if $s_i = +\infty$, then (19) is trivial. Fix $i \in I$, $s_i \neq 1$, and some positive integer $n' \leq n_0$. Let A be any $\xi_{n'}$ -measurable set. Then A and $\tilde{T}_{h_i} A$ are ξ_n -measurable. By definition,

$$\begin{aligned} \mu(\tilde{T}_{h_i}^{-1} A \Delta \tilde{T}_{\tilde{g}_i}^{-1} A) &= \mu(\tilde{T}_{h_i} A \Delta \tilde{T}_{\tilde{g}_i} A) = \\ 2\mu(\tilde{T}_{h_i} A \setminus \tilde{T}_{\tilde{g}_i} A) &= 2\mu((\tilde{T}_{h_i} A \cap B_0(i)) \setminus (\tilde{T}_{\tilde{g}_i} A \cap B_0(i))) = \\ 2\mu((\tilde{T}_{h_i} A \cap B_0(i)) \setminus (\tilde{T}_{h(i)} \tilde{T}_{\tilde{g}_1}^{k_1(i)} \dots \tilde{T}_{\tilde{g}_{i-1}}^{k_{i-1}(i)} \tilde{T}_{h_i}^{1-s_i} A \cap B_0(i))) &= \\ \frac{2}{s_i} \mu(A \setminus \tilde{T}_{h(i)} \tilde{T}_{\tilde{g}_1}^{k_1(i)} \dots \tilde{T}_{\tilde{g}_{i-1}}^{k_{i-1}(i)} \tilde{T}_{h_i}^{1-s_i} A) &= \\ \frac{1}{s_i} \mu(\tilde{T}_{h_i}^{s_i} A \Delta \tilde{T}_{h(i)} \tilde{T}_{\tilde{g}_1}^{k_1(i)} \dots \tilde{T}_{\tilde{g}_{i-1}}^{k_{i-1}(i)} A). \end{aligned}$$

Therefore,

$$d_{n'}(\tilde{T}_{\tilde{g}_i}, \tilde{T}_{h_i}) = \frac{1}{s_i} d_{n'}(\tilde{T}_{h_i}^{s_i}, \tilde{T}_{h(i)} \tilde{T}_{\tilde{g}_1}^{k_1(i)} \dots \tilde{T}_{\tilde{g}_{i-1}}^{k_{i-1}(i)}) \text{ for } n' \leq n_0.$$

Now if we recall (15), we get

$$d(\tilde{T}_{\tilde{g}_i}, \tilde{T}_{h_i}) < \frac{1}{s_i} d(\tilde{T}_{h_i}^{s_i}, \tilde{T}_{h(i)} \tilde{T}_{\tilde{g}_1}^{k_1(i)} \dots \tilde{T}_{\tilde{g}_{i-1}}^{k_{i-1}(i)}) + \frac{\gamma}{\lambda}. \quad (20)$$

Remark that (20) is also true for $s_i = 1$. Taking into account the commutativity, we then have

$$\begin{aligned} d(\tilde{T}_{h(i)} \tilde{T}_{\tilde{g}_1}^{k_1(i)} \dots \tilde{T}_{\tilde{g}_{i-1}}^{k_{i-1}(i)}, \tilde{T}_{h(i)} \tilde{T}_{h_1}^{k_1(i)} \dots \tilde{T}_{h_{i-1}}^{k_{i-1}(i)}) &\leq \\ \sum_{j < i} |k_j(i)| d(\tilde{T}_{\tilde{g}_j}, \tilde{T}_{h_j}) &\leq \sum_{j < i} c d(\tilde{T}_{\tilde{g}_j}, \tilde{T}_{h_j}) < \frac{2\gamma}{9(3c)^{r-i}}. \end{aligned} \quad (21)$$

Since $\tilde{T}' \in U$, by (17-8), we get

$$d(\tilde{T}_{h_i}^{s_i}, \tilde{T}_{h(i)} \tilde{T}_{h_1}^{k_1(i)} \dots \tilde{T}_{h_{i-1}}^{k_{i-1}(i)}) < d(T_{h_i}^{s_i}, T_{h(i)} T_{h_1}^{k_1(i)} \dots T_{h_{i-1}}^{k_{i-1}(i)}) + \frac{\gamma}{18(3c)^{r-i}}. \quad (22)$$

Besides,

$$d(T_{\tilde{g}_i}^{s_i}, T_{h(i)} T_{\tilde{g}_1}^{k_1(i)} \dots T_{\tilde{g}_{i-1}}^{k_{i-1}(i)}) = 0.$$

Therefore, applying (14), we get

$$\begin{aligned} d(T_{h_i}^{s_i}, T_{h(i)} T_{h_1}^{k_1(i)} \dots T_{h_{i-1}}^{k_{i-1}(i)}) &\leq s_i d(T_{\tilde{g}_i}, T_{h_i}) + \\ \sum_{j < i} |k_j(i)| d(T_{\tilde{g}_j}, T_{h_j}) &< \frac{\gamma s_i}{\lambda}. \end{aligned} \quad (23)$$

Combining (21-3), we obtain

$$\frac{1}{s_i} d(\tilde{T}_{h_i}^{s_i}, \tilde{T}_{h(i)} \tilde{T}_{\tilde{g}_1}^{k_1(i)} \dots \tilde{T}_{\tilde{g}_{i-1}}^{k_{i-1}(i)}) < \frac{\gamma}{\lambda} + \frac{5\gamma}{18(3c)^{r-i}} < \frac{11\gamma}{36(3c)^{r-i}}.$$

Because of (20), then (19) is proved.

Keeping in mind (19), (16), (14), we get,

$$d(\tilde{T}_{\tilde{g}_i}, T_{\tilde{g}_i}) \leq d(\tilde{T}_{\tilde{g}_i}, \tilde{T}_{h_i}) + d(\tilde{T}_{h_i}, T_{h_i}) + d(T_{h_i}, T_{\tilde{g}_i}) < \frac{\gamma}{3} + \frac{\gamma}{\lambda} + \frac{\gamma}{\lambda} < \gamma$$

for any i , and Theorem 4.1 follows. \square

Fix some finite measurable partition $\xi = \{C_1, \dots, C_m\}$ onto the sets of equal measure (not necessarily $\cup_i C_i = X$, but the difference will not be anyhow essential in the sequel).

Definition 4.2. We say that a G -action P is H, ξ -finite for some $H \leq G$ if for any $g \in H$ $P_g \xi = \xi$, $(\exists i_0) P_g C_{i_0} = C_{i_0}$ implies P_g is the identity, and $P|_H$ is transitive on $X/\xi = \{1, \dots, m\}$.

It is easy to see that every H, ξ -finite action P is H -finite, and $\langle P_g : g \in H \rangle = \oplus_{i=1}^k \langle P_{g_i} \rangle$. Denote $p_i = \text{deg } P_{g_i}$ in Ω , $i = 1, \dots, k$, $p = (p_1, \dots, p_k)$. Mark the element C_1 , then a map $\varphi(P_g) = P_g C_1$ is an isomorphism between $\oplus_{i=1}^k \langle P_{g_i} \rangle$ and X/ξ . Thus we get a parametrization of X/ξ by elements $\sum c_i g_i, c_i = 0, \dots, p_i - 1, i = 1, \dots, k$.

Definition 4.3. We say that a G -action T admits a good approximation by finite actions if there exists a sequence of H_n, ξ_n -finite actions $P(n)$ satisfying both

$$\xi_n \rightarrow \varepsilon, \text{ and}$$

$$\omega_n^2 \sum_{g: g = \sum c_i g_i(n), |c_i| \leq 2p_i(n)} \mu(T_g C_1(n) \Delta P_g(n) C_1(n)) = o(1),$$

where $\omega_n = \#\xi_n = \prod_i p_i(n)$.

Remark 4.4. If a transformation admits a cyclic approximation by periodic transformations with speed $o(1/n^3)$ (see notation in [21], [10]), then the corresponding \mathbb{Z} -action admits a good approximation by finite actions.

Besides, it is easy to see that any action of any finite group can not admit a good approximation by finite actions.

The proof of the following theorem has an independent interest, because it provides us by some constructive information about approximation sequences to elements of centralizers $C\{T_g : g \in G\}$ for a typical group action T .

Theorem 4.5. *If an action T of an infinite countable abelian group G admits a good approximation by finite actions, then*

$$C\{T_g : g \in G\} = CL\{T_g : g \in G\}, \quad (24)$$

and the set of all G -actions of any infinite countable abelian group G admitting a good approximation by finite actions forms a typical set.

What we really need in the main theorem is just (24) for a typical group action, and this is well-known for $G = \mathbb{Z}$. Nowadays it is usually attributed to King (see [23]) exploiting the fact that rank 1 transformations are typical. However it was known before because of a primary proof (see [9]) relying on so called Chacon's lemma [8]. We do not pretend that the weak-closure theorem was unknown for any countable abelian group G , however we did not find in the literature any mention about that. We are only aware of [27] where it is proved for infinite torsion-free countable abelian groups. We will follow the primary proof applying the extension-to-groups technique developed in [1].

Proof. Take a transformation $S \in C\{T_g : g \in G\}$, assuming T admits a good approximation by (H_n-) finite actions $P(n)$. We will prove that for any measurable set A and n there exists $g(n) \in K(n) = \{\sum_i c_i g_i(n) : 0 \leq c_i < p_i(n)\}$ such that

$$\lim_{n \rightarrow \infty} \mu(SA \Delta P_{g(n)}(n)A) = 0. \quad (25)$$

Let us remark first that (25) implies (24). Indeed, for every $C_j(n)$ we can find $g' = g'(j) \in K(n)$ such that $C_j(n) = P_{g'}(n)C_1(n)$. Therefore

$$\begin{aligned} \mu(T_{g(n)}C_j(n) \Delta P_{g(n)}(n)C_j(n)) &\leq \mu(T_{g(n)+g'}C_1(n) \Delta P_{g(n)+g'}(n)C_1(n)) + \\ &\quad \mu(T_{g'}C_1(n) \Delta P_{g'}(n)C_1(n)). \end{aligned}$$

From now on we apply that $\rho(A, B) = \mu(A \Delta B)$ is a metric on the set of all the measurable sets in X if we consider them up to zero measure. Let $A(\xi_n)$ be a nearest to A ξ_n -measurable set. Obviously,

$$\rho\left(\bigcup_j B_j, \bigcup_j D_j\right) \leq \sum_j \rho(B_j, D_j)$$

for any collection of measurable sets B_j, D_j . Therefore,

$$\omega_n^2 \mu(T_{g(n)}A(\xi_n) \Delta P_{g(n)}(n)A(\xi_n)) = o(1),$$

and

$$\lim_{n \rightarrow \infty} \mu(SA \Delta T_{g(n)}A) = 0. \quad (26)$$

In order to get a sequence $g(n)$ that is independent of A , we modify ξ_n to $\xi'_n = \{C'_1(n), \dots, C'_{\omega_n}(n)\}$, where $C'_1(n) = C_1(n) \setminus \bigcup_j T_{g'(j)}^{-1}C_j(n)$, $C'_j(n) = T_{g'(j)}C'_1(n)$. Choosing a sequence $g(n)$ for $C'_1(k)$, we get (26) for any ξ'_k -measurable set A . Since

$\xi'_n \rightarrow \varepsilon$, by the diagonal process, we have a universal sequence $g(n)$ and (25) implies (24).

In order to prove (25), denote $C_g(n) = P_g(n)C_1(n)$, $g \in K(n)$. Then $\xi_n = \{C_g(n) : g \in K(n)\}$. We write $A \sim_\varepsilon B$ if $\rho(A, B) < \varepsilon$, $1B = B$, and $0B = \emptyset$ for any set B . Fix $\varepsilon > 0$. For n large enough we have

$$\begin{aligned} A \sim_\varepsilon A(\xi_n) &= \bigsqcup_{g \in K(n)} a_g(n)C_g(n), \\ SA \sim_\varepsilon SA(\xi_n) &= \bigsqcup_{g \in K(n)} a'_g(n)C_g(n), \end{aligned}$$

where $a_g(n), a'_g(n) \in \{0, 1\}$, and

$$S(A(\xi_n)) = \bigsqcup_{g \in K(n)} a_g(n)SC_g(n) \sim_{\delta_1} \bigcup_{g \in K(n)} a_g(n)T_gSC_1(n),$$

here $\delta_1 \leq \sum_{g \in K(n)} a_g(n)\mu(T_gC_1(n)\Delta P_g(n)C_1(n))$. Therefore, for n large enough we obtain

$$SA(\xi_n) \sim_\varepsilon \bigcup_{g \in K(n)} a_g(n)T_gSC_1(n).$$

Let

$$M_2(n) = \{x \in C_1(n) : T_gx \in P_g(n)C_1(n), g = \sum c_i g_i, \min_i(2p_i(n) - |c_i|) \geq 0\}.$$

It is clear that

$$\lim_{n \rightarrow \infty} \omega_n^2 \mu(M_2(n)\Delta C_1(n)) = 0. \quad (27)$$

Denote

$$\begin{aligned} B_g(n) &= \{x \in C_1(n) : Sx \in C_g(n)\}, \\ E_g(n) &= T_{-g}SB_g(n), \quad D(n) = \sqcup C_g(n). \end{aligned}$$

From now on indexes g in $C_g(n)$ and $a_g(n), a'_g(n)$ will be considered by "modulo $p(n) = (p_1(n), \dots)$ ". Because of (27), for n large enough, we have

$$\max_{g, g_1 \in K(n)} \omega_n^2 \mu(T_{g_1}C_g(n)\Delta C_{g_1+g}(n)) < \varepsilon, \quad \max_{g_1 \in K(n)} \omega_n \mu(T_{g_1}D(n)\Delta D(n)) < \varepsilon.$$

Therefore,

$$\begin{aligned} \bigcup_{g \in K(n)} a_g(n)T_g(SC_1(n) \setminus D(n)) &\sim_\varepsilon \\ \bigcup_{g \in K(n)} a_g(n)T_gSC_1(n) \setminus D(n) &\sim_\varepsilon SA(\xi_n) \setminus D(n) \sim_\varepsilon \emptyset. \\ SA(\xi_n) &\sim_{4\varepsilon} \bigcup_{g \in K(n)} a_g(n)T_gS \bigsqcup_{g_1} B_{g_1}(n) = \\ \bigcup_{g, g_1 \in K(n)} a_g(n)T_{g+g_1}E_{g_1}(n) &\sim_{\delta_2(n)} \bigcup_{g, g_1 \in K(n)} a_{g-g_1}(n)T_gE_{g_1}(n). \end{aligned} \quad (28)$$

It is easy to see that

$$\delta_2(n) < \omega_n^2 \max_{g, g'_1} \mu(E_g(n)\Delta T_{g'_1}E_g(n)),$$

where

$$g'_1 \in \{g \in H_n : g = \sum c_i g_i(n), (\forall i) c_i \in \{0, p_i(n)\}\}.$$

Let

$x \in \tilde{E}_g(n) = M_2(n) \cap T_{-g}S(B_g(n) \cap M_2(n)) \subseteq E_g(n)$,
and $y = S^{-1}T_g x$. Since $Sy \in T_g M_2(n)$, then $T_{-g'_1}y \in B_g(n)$. Hence $T_{-g'_1}x = T_{-g}ST_{-g'_1}y \in E_g(n)$, then $\tilde{E}_g(n) \subseteq T_{g'_1}E_g(n)$. Since

$$\lim_n \max_g w_n^2 \mu(\tilde{E}_g(n) \Delta E_g(n)) = 0,$$

we get

$$\lim_n \delta_2(n) = 0.$$

Taking into account (28), for n large enough we obtain

$$SA \sim_\varepsilon \bigsqcup_g a'_g(n) C_g(n) \sim_\varepsilon \bigsqcup_g T_g(\bigsqcup_{g_1} a_{g-g_1}(n) \tilde{E}_{g_1}(n)).$$

Since

$$T_g(\bigsqcup_{g_1} a_{g-g_1}(n) \tilde{E}_{g_1}(n)) \subseteq C_g(n),$$

then for $g \in F(n) = \{g \in K(n) : a'_g(n) = 1\}$ we get

$$C_g(n) \sim_{\beta_g} T_g(\bigsqcup_{g_1} a_{g-g_1}(n) \tilde{E}_{g_1}(n)) \sim_{\gamma_g} \bigcup_{g_1} a_{g-g_1}(n) T_g E_{g_1}(n). \quad (29)$$

$$SA \sim_\varepsilon \bigsqcup_{g \in F(n)} C_g(n), \quad (30)$$

where

$$\max\left\{\sum_{g \in F(n)} \beta_g, \sum_{g \in F(n)} \gamma_g\right\} < \varepsilon.$$

Let us remaind the following combinatorial lemma.

Lemma 4.6. (Chacon [8]) *Let $\|x_{ji}\|, j = 1, \dots, k, i = 1, \dots, n$ be a matrix with 0, 1 entries, and $b_j, j = 1, \dots, k$ be nonnegative numbers, $\sum_j b_j = 1$. If there is $H \subseteq \{1, \dots, n\}$ such that*

$$\sum_j b_j x_{ji} \geq 1 - \eta \text{ for any } i \in H,$$

where $\eta \in (0, 1)$, then there exists a γ ($1 \leq \gamma \leq k$) such that

$$\sum_{i \in H} \sum_j b_j x_{ji} x_{\gamma i} \geq \#H(1 - 2\eta).$$

Fix some $\eta \in (0, 1)$. Let

$$b_{g_1} = \mu(\tilde{E}_{g_1}(n)) / \sum_g \mu(\tilde{E}_g(n)),$$

$$H(n) = \{g \in F(n) : \sum_{g_1} a_{g-g_1}(n) b_{g_1} \geq 1 - \eta\},$$

$$I(n) = F(n) \setminus H(n).$$

The sets $\tilde{E}_g(n)$ are mutually disjoint, therefore from (29) and (30) we have

$$\lim_n \mu\left(\bigsqcup_{g \in I(n)} C_g(n)\right) = 0,$$

$$SA \sim_\varepsilon \bigsqcup_{g \in H(n)} T_g \bigsqcup_{g_1} a_{g-g_1}(n) \tilde{E}_{g_1}(n) \sim_\varepsilon \bigsqcup_{g \in H(n)} C_g(n). \quad (31)$$

Applying Lemma 4.6 for $x_{g_1g} = a_{g-g_1}(n)$, we find a $g_0 \in K(n)$ such that

$$\sum_{g \in H(n)} \sum_{g_1} b_{g_1} x_{g_1g} x_{g_0g} \geq \#H(n)(1-2\eta), \text{ i.e.} \quad (32)$$

$$\sum_{g \in H(n)} \sum_{g_1} \mu(T_g \tilde{E}_{g_1}(n)) a_{g-g_1}(n) a_{g-g_0}(n) \geq \#H(n)(1-2\eta) \sum_g \mu(\tilde{E}_g(n)).$$

Take $\eta = \varepsilon/2$. It is clear that

$$\bigsqcup_{g \in H(n)} T_g \bigsqcup_{g_1} a_{g-g_1}(n) \tilde{E}_{g_1}(n) \supseteq \bigsqcup_{g \in H(n)} a_{g-g_0}(n) T_g \bigsqcup_{g_1} a_{g-g_1}(n) \tilde{E}_{g_1}(n).$$

Combining with (31-2), for n large enough we obtain

$$SA \sim_\varepsilon \bigsqcup_{g \in H(n)} a_{g-g_0}(n) \bigsqcup_{g_1} a_{g-g_1}(n) T_g \tilde{E}_{g_1}(n).$$

Combining with (29), we get

$$SA \sim_{2\varepsilon} \bigsqcup_{g \in H(n)} a_{g-g_0}(n) C_g(n) = \bigsqcup_{g: g+g_0 \in H(n)} a_g(n) C_{g+g_0}(n) =$$

$$P_{g_0}(n) \bigsqcup_{g: g+g_0 \in H(n)} a_g(n) C_g(n) \subseteq P_{g_0}(n) \bigsqcup_g a_g(n) C_g(n) = P_{g_0}(n) A(\xi_n).$$

It implies that for n large enough we have

$$SA \sim_\varepsilon P_{g_0}(n) A,$$

and (25) is proved.

In order to prove the final part of Theorem 4.5, pick a sequence of positive numbers $\varepsilon_n \rightarrow 0$ as $n \rightarrow +\infty$, and $G_k = \langle g_1, \dots, g_k \rangle$, where $G = \langle g_1, \dots, g_n, \dots \rangle$. Consider the set $L_{n,k}'^*$ as in Lemma 2.3. Obviously, every $P \in L_{n,k}'^*$ is G_k, ξ_n -finite. Fix some presentation $\langle P_g : g \in G_k \rangle = \bigoplus_{i=1}^l \langle P_{g_i(n)} \rangle$, $p_i(n) = \deg P_{g_i(n)}$, $i = 1, \dots, l$, $C_1(n)$, and $\omega_n = \prod_i p_i(n)$. Denote

$$U(P) = \{T \in \Omega_G : \omega_n^2 \sum_{g: g = \sum c_i g_i(n), |c_i| \leq 2p_i(n)} \mu(T_g C_1(n) \Delta P_g C_1(n)) < \varepsilon_n\}.$$

$$B_{n_0} = \bigcup_{n \geq n_0} \bigcup_k \bigcup_{P \in L_{n,k}'^*} U(P), \quad B = \bigcap_{n_0} B_{n_0}.$$

Since

$$\bigcup_{n \geq n_0} \bigcup_k L_{n,k}'^* \subseteq B_{n_0},$$

applying Lemma 2.3, we get that every B_{n_0} is a dense open set in Ω_G . Therefore B is a dense G_δ -set and consists of G -actions admitting a good approximation by finite actions. Theorem 4.5 is proved. \square

5. EXAMPLES

Take some prime number p . Let $\mathbb{C}_{p^\infty} = \mathbb{Q}_p/\mathbb{Z}$, where \mathbb{Q}_p stands for the additive group of all rational numbers of the form i/p^n , $i, n \in \mathbb{Z}$. Consider a pair $(\mathbb{C}_{p^\infty}, G)$, where \mathbb{C}_{p^∞} is a fixed subgroup of an abelian countable group G .

Example 5.1. of a free G -action T satisfying

$$CL\{T_g : g \in G\} = CL\{T_h : h \in \mathbb{C}_{p^\infty}\}.$$

Since \mathbb{C}_{p^∞} is **divisible**, i.e. for every its element a and n there is $b \in \mathbb{C}_{p^\infty}$ such that $nb = a$, G can be represented as $G = \mathbb{C}_{p^\infty} \oplus G^*$ for some G^* . It is well enough to construct such G -action T acting by shifts Q_x on the infinite-dimensional torus $X = (\mathbb{R}/\mathbb{Z})^\mathbb{N}$ equipped with the Haar measure, where $Q_x y = x + y$, $x, y \in X$. Fix some splitting of indexes $\mathbb{N} = I_1 \sqcup I_2$, where $\#I_1 = \#I_2 = \#\mathbb{N}$. Denote $x'_i(n) = 0$ if $i \in I_1$, and $x'_i(n) = 1/p^n$ if $i \in I_2$, $x'(n) = (x'_1(n), \dots, x'_k(n), \dots) \in X$, $T'_{ig_n} = Q_{x'(n)}$, $g_n = 1/p^n \in \mathbb{C}_{p^\infty}$, $n, i \in \mathbb{N}$. Then T' is a well-defined \mathbb{C}_{p^∞} -action. Let us modify T' to a \mathbb{C}_{p^∞} -action T we need. Fix a countable base of open cylindric sets $U_i = \{x \in X : x_{k_j} \in \Delta_j, j = 1, \dots, m\}$, here Δ_j are open intervals in \mathbb{R}/\mathbb{Z} . Let $n(i) = \max_j k_j$. Making no loss in the generality, we can assume that $n(1) < n(2) < \dots$. Consider the homomorphism P of X defined by

$$P(x_1, \dots, x_k, \dots) = (px_1, \dots, px_k, \dots).$$

It is clear that the set of all the pre-images of x , i.e. $\{P^{-n}x : n \in \mathbb{N}\}$ is dense in X for any x . Therefore there exist $m(1) \in \mathbb{N}$, $\tilde{x}(1) \in U_1$ such that $P^{m(1)}\tilde{x}(1) = 0$. Put $x_i(m(1)) = \tilde{x}_i(1)$ if $i \leq n(1)$, $x_i(m(1)) = x'_i(m(1))$ if $i > n(1)$, $x(m(1)) = (x_1(m(1)), \dots)$. Then $x(m(1)) \in U_1$, $P^{m(1)}x(m(1)) = 0$, and $x(m(1)) = x'(m(1))$ up to finitely many coordinates. Step by step, by the same argument we can find a sequence of $x(m(k)) \in X$, $m(k) - m(k-1) \in \mathbb{N}$, $k = 2, 3, \dots$ such that $x(m(k)) \in U_k$, $P^{m(k) - m(k-1)}x(m(k)) = x(m(k-1))$, and $x(m(k)) = x'(m(k))$ up to finitely many coordinates. Denote $T_{g_{m(k)}} = Q_{x(m(k))}$. Then every shift Q on X is contained in $CL\{T_{g_{m(k)}} : k \in \mathbb{N}\}$, and

$$T_{g_{m(k)}}^{p^{m(k) - m(k-1)}} = T_{g_{m(k-1)}}, \text{ and } T_{g_{m(k)}}^{p^{m(k)}} \text{ is the identity map.}$$

This means that T can be extended to a \mathbb{C}_{p^∞} -action T .

Let us finally define $T|_{G^*}$. Take any free G^* -action \tilde{T} acting by shifts on $\times_{i \in I_1} \mathbb{R}/\mathbb{Z}$. Then $T|_{G^*}$ on $X = \times_{i \in I_1} \mathbb{R}/\mathbb{Z} \times \times_{i \in I_2} \mathbb{R}/\mathbb{Z}$ is just the diagonal action $\tilde{T} \times E$, where E is the G^* -action on $\times_{i \in I_2} \mathbb{R}/\mathbb{Z}$ by the identities.

Obviously, $T|_{G^*}$ consists of shifts on X . Therefore T extends to a G -action by shifts, and

$$CL\{T_g : g \in G\} = CL\{T_{g_{m(k)}} : k \in \mathbb{N}\} = CL\{T_h : h \in \mathbb{C}_{p^\infty}\}.$$

Let $T_{g'}y = y$ for some $y \in X, g' \in G$. Then $x(g') = 0$. By definition, if $g = i/p^k \in \mathbb{C}_{p^\infty}$, then $x_j(g) = i/p^k$ for "almost all" $j \in I_2$, and if $g \in G^*$, then $x_j(g) = 0$ for all $j \in I_2$. Therefore, $g' \in G^*$, $\tilde{T}_{g'}$ is the identity, $g' = 0$, and the freeness of T follows.

Let $H^* = \bigoplus_{i=1}^\infty \langle h_i \rangle$, where $\deg h_i = p_i \rightarrow +\infty$ as $i \rightarrow +\infty$. Consider a pair (H^*, G) , where H^* is a fixed subgroup of an abelian countable group G .

Example 5.2. of a free G -action T satisfying

$$CL\{T_g : g \in G\} = CL\{T_h : h \in H^*\}.$$

It is clear that we can find a sequence $g_i \in G, i \in \mathbb{N}$ such that $G = \langle g_i : i \in \mathbb{N} \rangle$ and for some sequence $k_i, g_{k_i} \in H^*$, $\deg g_{k_i} \rightarrow +\infty$ as $i \rightarrow +\infty$, and $G_{k_i} = G_{k_{i-1}} \oplus \langle g_{k_i} \rangle$, where $G_k = \langle g_1, \dots, g_k \rangle$. Indeed, it follows from the fact that every abelian finitely generated group has the finite torsion part.

Consider some splitting of the indexes $\mathbb{N} = \sqcup_{i \in \mathbb{N}} I_i$, where for any $i \notin I_i = \emptyset$. Let $\{\alpha_i\}, i \in \mathbb{N}$ be any sequence of irrational numbers. Denote $(t_1, t_2, \dots) = (1, 1, 2, 1, 2, 3, 1, \dots)$.

Assume that at the step n a free G_n -action T acting by shifts Q_x on $X = (\mathbb{R}/\mathbb{Z})^{\mathbb{N}}$ was constructed, where for all $x(g) = (x_1(g), \dots), g \in G_n, x_i(g) = 0$ if $i \in \cup_{m > n} I_m \setminus J_n$ for some finite $J_n \subset \mathbb{N}$. Let us define a shift $T_{g_{n+1}}$ as follows. If for any $i, n+1 \neq k_i$, then denote

$$m = \min_{\substack{sg_{n+1} \in G_n \\ s > 0}} s,$$

or $m = +\infty$ if for any $s > 0, sg_{n+1} \notin G_n$. If $m = +\infty$, then put $x_i(g_{n+1}) = \alpha_{n+1} \pmod{1}$ for $i \in I_{n+1}$, and $x_i(g_{n+1}) = 0$ for $i \notin I_{n+1}$. If $m \neq +\infty$ then $mg_{n+1} = g_0 \in G_n$, and we put $x_i(g_{n+1}) = 1/m \pmod{1}$ for $i \in I_{n+1}$, and $x_i(g_{n+1}) = x_i(g_0)/m$ for $i \notin I_{n+1}$. Finally, if $n+1 = k_i$, then put $x_i(g_{n+1}) = 1/d$ for $i \in (I_{n+1} \cap \{j > t_i\}) \cup \{t_i\}$, and $x_i(g_{n+1}) = 0$ otherwise, here $\deg g_{k_i} = d$.

It is easy to see that, in any case, we get a well-defined free G_{n+1} -action satisfying $x_i(g) = 0$ if $g \in G_{n+1}$ and $i \in \cup_{m > n+1} I_m \setminus J_{n+1}$ for some finite $J_{n+1} \subset \mathbb{N}$. This means that we defined a free G -action T . Besides, by the above construction, for any pair of positive integers m, n there exists $h(m, n) \in H^*$ such that $\deg h(m, n) = d(m, n) > m$, $x_n(h(m, n)) = 1/d(m, n)$, and $x_i(h(m, n)) = 0, i = 1, \dots, n-1$. It follows that every shift is contained in $CL\{T_h : h \in \langle h(m, n) : m, n \in \mathbb{N} \rangle\}$. Thus, we get

$$CL\{T_g : g \in G\} = CL\{T_h : h \in H^*\}.$$

Consider a pair (H^{**}, G) , where H^{**} is a fixed subgroup of $G = \oplus_{i=1}^{\infty} \mathbb{Z}/m\mathbb{Z}$ isomorphic to $G, m \in \mathbb{N}$.

Example 5.3. of a free G -action T satisfying

$$CL\{T_g : g \in G\} = CL\{T_h : h \in H^{**}\}.$$

Since the subgroup H^{**} is **servant**, i.e. any equation $nx = a, a \in H^{**}, n \in \mathbb{N}$ that is solvable in G is solvable in H^{**} as well, G can be represented as $G = H^{**} \oplus G^*$ for some G^* . Construct such G -action T acting by shifts Q_x on $X = (\mathbb{R}/\mathbb{Z})^{\mathbb{N}}$, where $Q_x y = x + y, x, y \in X$. Fix some splitting of indexes $\mathbb{N} = I_1 \sqcup I_2$, where $\#I_1 = \#I_2 = \#\mathbb{N}$. Take any free G^* -action \tilde{T} acting by shifts on $\times_{i \in I_1} \mathbb{R}/\mathbb{Z}$. Then $T|_{G^*}$ on $X = \times_{i \in I_1} \mathbb{R}/\mathbb{Z} \times \times_{i \in I_2} \mathbb{R}/\mathbb{Z}$ is the diagonal action $\tilde{T} \times E$, where E is the G^* -action on $\times_{i \in I_2} \mathbb{R}/\mathbb{Z}$ by the identities. Denote $x'_i(n) = 1/m$ if $i = n$, and $x'_i(n) = 0$ if $i \neq n, x'(n) = (x'_1(n), \dots) \in X$. Pick a map $\varphi : I_2 \rightarrow G^*$ such that for any $g \in G^*, \#\varphi^{-1}(g) = \#\mathbb{N}$. We can assume that $H^{**} = \oplus_{i \in I_2} \langle h_i \rangle$, where $\deg h_i = m, i \in I_2$. Put

$$T_{h_i} = Q_{x(\varphi(i)) + x'(i)} = T_{\varphi(i)} Q_{x'(i)}, i \in I_2.$$

It is easy to see that we got a well-defined G -action. Besides, by definition, for any $g \in G^*$ there exists $i_n \rightarrow +\infty$, such that $\varphi(i_n) = g$. Therefore

$$T_{h_{i_n}} = T_g Q_{x'(i_n)} \rightarrow T_g \text{ as } n \rightarrow +\infty,$$

and T is the G -action as it is required.

6. PROOF OF THE MAIN THEOREM AND APPLICATIONS

Every bounded abelian group G is uniquely represented as the direct sum of finitely many p_i -groups G_{p_i} , where p_i are mutually different prime numbers, and $G_p = \{g \in G : \exists k p^k g = 0\}$. Moreover, by the classical Prüfer theorem every such G_{p_i} can be represented as the direct sum of cyclic subgroups. Calculating multiplicities of summands of equal order, we get a finitely supported function, noted

$$m_G : \{p^k : p \text{ is prime and } k > 0\} \rightarrow \mathbb{N} \cup \{0, +\infty\},$$

which is independent of our choice of representations G_{p_i} as the direct sums.

It is clear that two bounded countable abelian groups G_+ and G_- are isomorphic iff $m_{G_+} = m_{G_-}$. This implies in particular that every bounded infinite countable abelian group is not cohopfian.

Consider \overline{m}_G defined by

$$\overline{m}_G(p^k) = \sum_{n \geq k} m_G(p^n), p \text{ is prime and } k > 0.$$

Then $H \leq G$ implies $\overline{m}_H \leq \overline{m}_G$. It follows that two bounded countable abelian groups G_+ and G_- are weakly isomorphic if and only if $\overline{m}_{G_+} = \overline{m}_{G_-}$.

Let us prove the main theorem.

Proof. The case H is finite is trivial. Namely, the well-known Bernoulli action T on, say, $\{0, 1\}^G$ equipped with the Haar measure acts freely by shifts. Thus $T|_H$ is free. However all free H -actions are pairwise isomorphic and form a typical set. This means that a typical H -action can be extended to a free Bernoulli G -action.

Let H contain an isomorphic copy of \mathbb{Z} . Consider a free G -action T as in Example 3.1. We have

$$CL\{T_g : g \in G\} = CL\{T_h : h \in H\} = CL\{T_h : h \in \mathbb{Z}\},$$

because the centralizer of ergodic transformation with discrete spectrum is just the closure of its powers. By Theorem 4.1, T is a locally-dense point for π_H . It is well known that all the conjugate actions to any fixed free action are dense for any amenable group G . This means that we have a dense set of locally dense points for π_H . By the same argument as in the proof of Theorem 3.3, we have that a typical H -action can be extended to a free G -action.

Let H be an infinite torsion group. If there exists a subgroup $H^* = \oplus_{i=1}^{\infty} \langle h_i \rangle$, where $\deg h_i \rightarrow +\infty$ as $i \rightarrow +\infty$, then we are done by the same argument based on Example 5.2. Besides, it is easy to see that H has such subgroup H^* if and only if for any m, n there exists a finitely generated subgroup, noted H^0 , admitting some presentation $H^0 = \oplus_{i=1}^N \langle h'_i \rangle$, where $\#\{i : \deg h'_i > m\} > n$. This means that no such subgroup H^* for H implies that for some positive integer m mH has finite rank. It is well known that an abelian torsion group has finite rank if and only if it is a direct sum of finitely many finite cyclic groups and \mathbb{C}_{p^∞} . If there exists some $\mathbb{C}_{p^\infty} \leq mH \leq H$, then, applying Example 5.1, we get what we

claimed. If not, then there exists m' such that $\#m'H = 1$. In this case H can be represented as $H = \bigoplus_{j=1}^s H_{p_j}$, where p_j are mutually different prime numbers and $H_p = \{h \in H : (\exists k > 0) \deg h = p^k\}$. Moreover, $H_{p_j} = \bigoplus_{i < h_i(j) > \text{ for any } j$, where $\deg h_i(j) | m'$ for any i, j . Denote

$$d_j = \max k : \#\{i : \deg h_i(j) = k\} = \#N,$$

and $d_j = 1$ if $\#H_{p_j}$ is finite, $j = 1, \dots, s$. Let $d = \prod_j d_j$ and

$$\begin{aligned} H_{p_j}^> &= \bigoplus_{i: \deg h_i(j) > d_j} < h_i(j) >, \quad H_{p_j}^{\leq} = \bigoplus_{i: \deg h_i(j) \leq d_j} < h_i(j) >, \\ H_{p_j}^d &= \bigoplus_{i: \deg h_i(j) = d_j} < h_i(j) >. \end{aligned}$$

It is clear that $H^> = \bigoplus_j H_{p_j}^>$ is finite and $H = H^> \oplus H^{\leq}$, where $H^{\leq} = \bigoplus_j H_{p_j}^{\leq}$.

Let G be any abelian countable group ($H \leq G$). Assume that every $g \in G$ admits a representation $g = h + h_1$ for some $h \in H^>$, $dh_1 = 0$. Then $G = \bigoplus_{j=1}^s G_{p_j}$ and $H_{p_j}^>$ is servant in G_{p_j} for any j . Therefore G can be represented as

$$G = H^> \oplus G^* \text{ for some countable abelian } G^*, \#dG^* = 1. \quad (33)$$

It is clear that $H = H^> \oplus H'$ for some subgroup $H' \leq G^*$, and $H' \cong H/H^> \cong H^{\leq}$. Fix a $G^{**} = \bigoplus_{i=1}^{\infty} < h'_i >$ such that $\deg h'_i = d$, and $G^* \leq G^{**}$. Then we can find a subgroup, noted H^d , of H' , which is isomorphic to $\bigoplus_j H_{p_j}^d \cong \bigoplus_i^{\infty} \mathbb{Z}/d\mathbb{Z}$. By Example 5.3, there exists a free G^* (and G^{**}) - action T' such that $CL\{T'_g : g \in G^*\} = CL\{T_h : h \in H'\}$. Let T'' be any free $H^>$ -action on a non-atomic standard Borel probability space (Y, \mathcal{F}_1, ν) . Then the cartesian product $T = T' \times T''$ acting on $(X \times Y, \mathcal{F} \otimes \mathcal{F}_1, \mu \otimes \nu)$ is a free G -action, and $CL\{T_g : g \in G\} = CL\{T_h : h \in H\}$. Therefore T is a locally dense point for π_H , and a typical H -action can be extended to a free G -action.

If G is still a countable abelian group, $H \leq G$, and is not as in (33), then we fix $g \in G$ that can not be represented as $g = h + h^*$, $h \in H^>$, $dh^* = 0$. Besides, by Theorem 4.5, for a typical H -action T , for any $S \in C\{T_h : h \in H\}$, we can find $h_i = h_i^> + h_i^{\leq}$ such that $T_{h_i} \rightarrow S$ as $i \rightarrow \infty$. Taking a subsequence, we have $T_{h_i^{\leq}} \rightarrow R = ST_{h^>}^{-1}$, and R^d is the identity. Assuming that there exists a G -action T extending H -action T , we obtain that $T_{h_i^{\leq}} \rightarrow R = T_g T_{h^>}^{-1}$ for some $h^> \in H^>$, $h_i^{\leq} \in H^{\leq}$, $i = 1, \dots$. Therefore $R = T_{h^*}$ for $h^* = g - h^>$, and $dh^* \neq 0$. Since R^d is the identity, the G -action T is not free. Therefore a typical H -action can not be extended to a free G -action.

Besides, it is easy to check that a countable abelian group G ($H \leq G$) satisfies (33) if and only if there exists some $G^{**} \geq G$ isomorphic to H . Theorem 1.2 is proved. \square

6.1. Applications.

Definition 6.1. Let X and Y be Polish spaces. We say that a (continuous) map $\varphi : X \rightarrow Y$ is (second/essential) category preserving if the images and the pre-images of second category sets are of the second category as well.

Obviously, if the map φ is ess. category preserving, then $\varphi(X)$ is a typical set, i.e. $Y \setminus \varphi(X)$ is meager. Besides, every pre-image (not image!) of a meager set is a meager set for ess. category preserving maps.

Let us remark that for every countable group G the space Ω_G has the *weak Rokhlin property*, i.e. there is a G -action T such that the set of all G -actions that are isomorphic to T is dense in Ω_G (see [14], [22]). This implies that for any amenable subgroup H ($H \leq G$) $\pi_H(\Omega_G)$ is dense in Ω_H^2 .

We note that if both $\varphi(X)$ is dense in Y and $\varphi(B)$ is of the second category for every second category set B , then φ is *ess. category preserving*. Indeed, in this case, for any countable intersection $\cap_i B_i$ of open dense sets B_i , $\varphi(\cap_i B_i)$ (as an analytic set) is an open set, say Y' , up to a meager set. It is easy to see that Y' is dense in Y . Therefore φ maps any typical set onto a typical set. Fix some set $C \subseteq Y$ of the second category. Assume $\varphi^{-1}(C)$ is meager; then $X \setminus \varphi^{-1}(C)$ is typical in X , and $\varphi(X \setminus \varphi^{-1}(C))$ is typical in Y . However $\varphi(X \setminus \varphi^{-1}(C)) \cap C = \emptyset$, and we have a contradiction. Therefore $\varphi^{-1}(C)$ is of the second category, and φ is *ess. category preserving*.

It was noticed in [2], Lemma 1, that if the map φ has a dense set of locally dense points, then $\varphi(B)$ is not meager for every non-meager B . This implies that φ is *ess. category preserving*. Therefore main theorem has the following application.

Theorem 6.2. *Let G be any countable abelian group and H its subgroup. Suppose H is not an infinite bounded group; then the restriction map $\pi_H : \Omega_G \rightarrow \Omega_H$ is *ess. category preserving*. Suppose H is an infinite bounded group; then π_H is *ess. category preserving* if and only if G is weakly isomorphic to H .*

Proof. Indeed, in fact, we proved that the set of locally dense points is dense for π_H and for any pair (H, G) except the case when H is a bounded subgroup and G is not weakly isomorphic to H . Besides, in remaining, by the main theorem, $\pi_H(F)$ is meager for every infinite H , where F stands for the set of all free G -actions. Therefore π_H is not *ess. category preserving*. And finally, let H be a finite subgroup of G . Assume π_H is not *ess. category preserving*. Then $\pi_H(A)$ is meager for some non-meager A . It clearly implies that $\pi_H(B)$ is meager for some non-empty open B . Fix a countable dense set, say C , in Ω . Let D be the union of all $\varphi^{-1}B\varphi$, $\varphi \in C$. Then $\pi_H(D)$ is meager. Besides, the set, say E , of all G -actions that are isomorphic to the Bernoulli G -action T is dense in Ω_G because T is free. It is easy to check both $E \subset D$ and $\pi_H(E)$ being the set of all free H -actions can not be meager. We then have a contradiction and Theorem 6.2 is proved. \square

Remark 6.3. According to [33], a continuous map respects the genericity if the images and the pre-images of typical sets are typical. Relying on the above discussion, it is easy to see that a continuous map is *ess. category preserving* if and only if it respects the genericity.

The reader can find a bit different look at relations between alternative conceptions of category preserving maps in [26], [33].

Remark 6.4. Let us illustrate how Theorem 6.2 works on a simple example of a dynamic property to be "not isomorphic to its inverse". It is well known (see [1]) that a typical \mathbb{Z}^d -action is not isomorphic to its inverse. Let G be any non-torsion countable abelian group. Take some $g \in G$ of infinite order. By Theorem 6.2 the

²This is also true for every non-amenable H as well, because we can correspond to every H -action T the so-called *co-induced* G -action \tilde{T} satisfying T is a factor (a quotient) of $\tilde{T}|_H$ (see [16], [22]). The density follows from a well-known fact that every factor of any G -action S is in the closure of conjugations of S if G is countable.

map $\pi_{<g>}$ is ess. category preserving. Therefore the set of not isomorphic to its inverse G -actions is a typical set.³

In the sequel, we keep the standard notation \widehat{G} for the group of characters of G , and $\mathbf{Ann} H$ for a subgroup $\{\chi \in \widehat{G} : (\forall h \in H)\chi(h) = 0\}$, where H is a subset of an abelian group G .

Theorem 6.5. *Let H be any infinite subgroup of a countable abelian group G . Then the following assertions are equivalent each other:*

- (i): *For a typical H -action there is an extension to a free G -action.*
- (ii): *There is a dense subgroup $G_1 \leq \widehat{G}$ such that $\#G_1 \cap \mathbf{Ann} H = 1$.*
- (iii): *There is a free (ergodic) G -action T with discrete spectrum such that $T|_H$ is ergodic.*

Let me leave an ergodic-theoretical proof of Theorem 6.5.

Proof. Suppose first that H is unbounded. Following the proof of the main theorem, we construct a free G -action T acting by shifts on the infinite-dimensional torus X . Moreover, the set of shifts corresponding to the H -subaction forms a dense set among all the shifts on X . It gives (iii). Let G_1 be the discrete part $\Lambda(T)$ of the spectrum of T , i.e. the set of all eigenvalues of G -action T . Then G_1 is a countable subgroup of \widehat{G} . The freeness of T implies the density of G_1 . Besides, if $\lambda \in G_1 \cap \mathbf{Ann} H$, then the corresponding eigenfunction f_λ is invariant for $T|_H$. Because of the ergodicity, f_λ is a constant function, and λ is the trivial character. It gives (ii).

We can apply the same argument for any infinite bounded H if it is weakly isomorphic to G . The only difference is the set of shifts corresponding to the H -subaction forms a dense set among all the shifts $T_g, g \in G$ on the infinite-dimensional torus X . Restricting ourselves to an ergodic component for $T|_H$, we get a $T|_G$ -invariant closed subset of X , which is just the closure of a $T|_H$ -orbit. Thus we have a free action as it is needed in (iii).

Therefore we get the implications (i) \Rightarrow (iii) \Rightarrow (ii).

Conversely, since H is infinite, G_1 is infinite. Fix an infinite countable dense subgroup $G_1^* \leq G_1$. Consider an ergodic G -action T with discrete spectrum, where $\Lambda(T) = G_1^*$. The density of G_1^* implies the freeness of T . Obviously, $\#G_1^* \cap \mathbf{Ann} H = 1$ implies $T|_H$ is an ergodic action with discrete spectrum. Applying the standard realization of every ergodic G -action S with discrete spectrum as a G -action by shifts on the character group $\widehat{\Lambda(S)}$ of the discrete part $\Lambda(S) \leq \widehat{G}$, it is easy to get a well-known fact that every transformation that commutes with S is just a shift on the abelian group $\widehat{\Lambda(S)}$. Therefore $CL\{T_g : g \in G\} = CL\{T_h : h \in H\}$. By Theorem 4.1, T is a locally-dense point for π_H . This means that we have a dense set of locally dense points for π_H . By the same argument as in the proof of Theorem 3.3, we have that a typical H -action can be extended to a free G -action. Therefore we proved that (ii) \Rightarrow (iii) \Rightarrow (i). \square

Remark 6.6. *A priori* Theorem 6.2 is not useful if π_H is not ess. category preserving, because the topological status of $\pi_H(B)$ can be essentially different for different

³In fact, applying the technique developed in [1], we can show that a typical G -action is not isomorphic to its inverse for any countable group G except obvious cases when G is finite or $G = \oplus^\infty \mathbb{Z}/2\mathbb{Z}$. However, the proof is, of course, much more complicated.

typical sets B . However the main theorem can be applied implicitly in this case as well (see Remark 6.11).

In order to proceed to that, let us describe all the pairs $H \leq G$ satisfying π_H is *extremely* non ess. category preserving, i.e. $\pi_H(B)$ is meager for every typical set B . By the 0 – 1 law, it is well enough to say when $\pi_H(\Omega_G)$ is a typical set.

Notice, first, that for every subgroup $G^* \leq G$ the space Ω_{G/G^*} can be viewed as a closed subset of Ω_G defined by the embedding $\varphi^* : \Omega_{G/G^*} \rightarrow \Omega_G$, where $\varphi^*(T)_g = T_{\varphi(g)}$, $g \in G, T \in \Omega_{G/G^*}$, φ is the canonical homomorphism of G onto G/G^* .

Theorem 6.7. *Let G be any countable abelian group and H its subgroup. Suppose H is not an infinite bounded group; then $\pi_H(\Omega_G)$ is a typical set. Suppose H is an infinite bounded group; then $\pi_H(\Omega_G)$ is a typical set if and only if there is a subgroup $G^* \leq G$ such that both $\#G^* \cap H = 1$ and G/G^* is weakly isomorphic to H . Moreover, if H is an infinite bounded group and there is such a subgroup G^* , then $\pi_H|_{\varphi^*(\Omega_{G/G^*})}$ is ess. category preserving.*

Proof. The first part follows from the main theorem. Let H be an infinite bounded group. " \Leftarrow " and the last part of the theorem is almost obvious. Indeed, $\varphi(H)$ is isomorphic to H via φ . Therefore

$$\pi_H(\varphi^*(\Omega_{G/G^*})) = \pi_{\varphi(H)}(\Omega_{G/G^*})$$

if we naturally identify Ω_H and $\Omega_{\varphi(H)}$. It remains to apply Theorem 6.2 for the map $\pi_{\varphi(H)}$.

In order to prove " \Rightarrow ", we keep the proof of the main theorem notation. It is easy to see that by Theorem 4.5, for a typical $T \in \Omega_H$, if there is a T extending G -action T then

$$\varphi_T(dG) = \varphi_T(dH^>) = \varphi_T(dH),$$

where φ_T is the homomorphism defined by $\varphi_T(g) = T_g, g \in G$. Since $\pi_H(\Omega_G)$ restricted to the set of free H -actions is a typical set as well, we can choose a free H -action T' admitting an extension to a G -action T' such that $\varphi_{T'}(dG) = \varphi_{T'}(dH)$. Denote $G^* = \mathbf{Ker} \varphi_{T'}$. Obviously, $\#G^* \cap H = 1$. Thanks to the homomorphism theorem, the group G/G^* is isomorphic to $\varphi_{T'}(G)$. Besides, H is isomorphic to $\varphi_{T'}(H)$. It remains to prove that $\varphi_{T'}(G)$ is weakly isomorphic to $\varphi_{T'}(H)$. However,

$$d\varphi_{T'}(G) = \varphi_{T'}(dG) = \varphi_{T'}(dH) = d\varphi_{T'}(H).$$

This completes the proof of Theorem 6.7. \square

A more explicit description of all pairs $H \leq G$ with $\pi_H(\Omega_G)$ is a typical set comes from the following theorem.

Theorem 6.8. *Let G be any countable abelian group, H its infinite bounded subgroup. Then $\pi_H(\Omega_G)$ is a typical set if and only if G can be represented as $H' \oplus G_1$, where $H \leq H' \leq G$, and H' is weakly isomorphic to H .*

This theorem implies that a typical H -action can be extended to a G -action because there is an extension to a free H' -action and every H' -action can be extended to a G -action by, for example, identities on G_1 .

Proof. " \Leftarrow " is a clear application of Theorem 6.7. To get " \Rightarrow ", we employ the proof of the main theorem notation. Fix a j , and a representation $H_{p_j} = H_{p_j}^> \oplus H_{p_j}^{\leq}$.

Remark first that $H_{p_j}^>$ is servant in G_{p_j} . If not, then consider the canonical homomorphism φ of G onto G/G^* , where G^* is as in Theorem 6.7. Therefore $\varphi(H_{p_j}^>)$ is not servant in G/G^* , because φ is the group isomorphism if it is restricted to H . However $\varphi(H_{p_j}^>)$ is the direct summand in G/G^* because G/G^* is weakly isomorphic to H . We get a contradiction.

By the the Prüfer-Kulikov theorem, for any countable abelian group, every its bounded servant subgroup is a direct summand. Thus $G_{p_j} = H_{p_j}^> \oplus G_{p_j}^*$, $H_{p_j} = H_{p_j}^> \oplus H_{p_j}^*$ for some $H_{p_j}^* \leq G_{p_j}^*$, where $H_{p_j}^*$ is isomorphic to $H_{p_j}^{\leq}$.

Let us define H_{p_j}' . If H_{p_j} is finite, then we put $H_{p_j}' = H_{p_j} = H_{p_j}^>$. Assume that H_{p_j} is infinite. Obviously, $\#d_j H_{p_j}^* = 1$, and $d_j = p_j^k$ for some positive integer k . Moreover, exploiting φ again, we get that

$$\#H_{p_j}^* \cap d_j G_{p_j}^* = 1.$$

Denote by M the non-empty set of all subgroups $H^* \leq G_{p_j}^*$ such that both $\#H^* \cap d_j G_{p_j}^* = 1$ and $H_{p_j}^* \leq H^*$. Consider the restriction to M of the natural partial order $G_1 \leq G_2$ on the set of all subgroups of G . Fix a maximal element, say H_{max}^* . Then H_{max}^* is servant in $G_{p_j}^*$ (see, for example, [12], Theorem 27.7). Obviously,

$$d_j H_{max}^* \subseteq H_{max}^* \cap d_j G_{p_j}^* = \{0\}, \text{ and } H_{p_j}^* \leq H_{max}^*.$$

Therefore H_{max}^* is weakly isomorphic to $H_{p_j}^*$.

Put $H_{p_j}' = H_{p_j}^> \oplus H_{max}^*$. It is clear that $H_{p_j} \leq H_{p_j}' \leq G_{p_j}$, H_{p_j}' is weakly isomorphic to H_{p_j} , and H_{p_j}' is a direct summand in G_{p_j} .

Let us finally define H' as $\oplus_j H_{p_j}'$. It is clear that H' is weakly isomorphic to H , and H' is a direct summand in the torsion part of G . Thus H' is a bounded servant subgroup of G , and Theorem 6.8 is proved. \square

Let us mention a dual analog of Theorem 6.8 as well.

Theorem 6.9. *Let G be any countable abelian group G , H its infinite subgroup. Then the following assertions are equivalent:*

- (i): *A typical H -action can be extended to a G -action.*
- (ii): *\hat{G} can be represented as the direct sum of two (possibly trivial) closed subgroups G_1^* and G_2^* such that both $\mathbf{Ann} H \geq G_2^*$ and there is a G_1^* -dense subgroup $G_1 \leq G_1^*$ satisfying $\#G_1 \cap \mathbf{Ann} H = 1$.*

This theorem is a natural corollary of Theorems 6.5, 6.8, and we leave the proof to the reader.

Remark 6.10. We omit certain dual versions of Theorem 6.7, or, say, of a slight modification of Theorem 6.8, where we replace the conditions on G by G can be represented as $G = G_1 \oplus G_2$ such that $\#G_2 \cap H = 1$ and G_1 is weakly isomorphic to H .

Remark 6.11. Assume H is an infinite bounded group and $\pi_H(\Omega_G)$ is a typical set. Denote

$$D = \{G^* \leq G : \#G^* \cap H = 1 \text{ \& } G/G^* \text{ is weakly isomorphic to } H\}.$$

In fact, proving Theorem 6.7 we showed the following claim.

$$\text{For a typical } C, \pi_H^{-1}(C) \subseteq A = \cup_{G^* \in D} \varphi^*(\Omega_{G/G^*}).$$

It implies that the question when $\pi_H(B)$ is a typical set is reduced to the same question for the restrictions of B to the subspaces $\varphi^*(\Omega_{G/G^*})$, where π_H becomes ess. category preserving.

In spite of the fact that D may have continuum many elements, the set A is always nowhere dense if G is not weakly isomorphic to H . Indeed, the density somewhere implies the density everywhere. Therefore, the set of locally dense points for π_H is dense in Ω_G , and π_H is ess. category preserving.

Besides, it is easy to check that

$$\cup_{G^* \in D} \varphi^*(\Omega_{G/G^*}) \subseteq B = \cap_g \cup_{h \in H} \{T \in \Omega_G : T_{dg} = T_{dh}\}.$$

We note that, as a rule, the inclusion $A \subseteq B$ is proper.

We now turn to another certain application of the main theorem. There is a bit stronger property then to be *monothetic* for topological groups. Namely, following [27], a topological group G is called *generically monothetic* if the set of g that generate a dense subgroup of G is a dense G_δ -set in G . Note that in the definition above, the condition that the set be G_δ is automatic, so one only has to check that it is dense. Note then that G is clearly generically monothetic if for any positive integer n $\langle ng_0 \rangle$ is dense in G for some $g_0 \in G$.

There has been considerable interest in the study of centralizers of typical group actions (see, for example, most new preprints [27], [31]). One of the new reasons for that, among others, is to understand all the similarities that come if we replace Ω , as a target for representations of a group, with other sufficiently well developed automorphism groups, say with the unitary group of a separable, infinite-dimensional Hilbert space or the group of isometries of the universal Urysohn metric space (see, for example, [27], [34]). Let us remind that the centralizer of a typical transformation is a closed abelian non-locally compact monothetic subgroup of Ω .

Theorem 6.12. *Let G be any countable abelian group. Then the centralizer of a typical G -action is generically monothetic if and only if G is unbounded.*

Proof. It is clearly true for any bounded G . Indeed, if G is finite, then the centralizer of every free G -action is just the group of all G -extensions under certain topology. Thus, it is not even abelian. Besides, for any infinite bounded G , by Theorem 4.5 every element of the centralizer is of finite order for a typical G -action. This means that the centralizer is not even (topologically) finitely generated.

Let G contain an isomorphic copy of \mathbb{Z} , say $\langle g \rangle$. Combining Theorems 6.2 and 4.5 for $H_n = \langle ng \rangle$, $n \in \mathbb{N}$, we easily deduce that

$$C\{T_h : h \in G\} = CL\{\langle T_{ng} \rangle, n \in \mathbb{N},$$

for a typical G -action T . Therefore, the centralizer is generically monothetic. It implies, in particular, that the centralizer is generically monothetic for a typical $H \oplus \mathbb{Z}$ -action, where H is any unbounded torsion abelian group. Applying Theorems 6.2 and 4.5 again, we have the same for a typical H -action. This completes the proof. \square

7. NON-ABELIAN CASE, QUESTIONS, AND COMMENTS

Let G be any countable group. The group Ω acts on Ω_G by conjugations. It is well known that the set of all G -actions with a dense orbit under conjugations, say $F' = F'_G$, forms a dense G_δ -set.

Since the set F of free G -actions is equal to F' for every countable amenable group G , the set F remains characteristic for the restriction map π_H in the amenable world according to the following fact.

Proposition 7.1. *Let H be any subgroup of a countable group G . The following are equivalent:*

- (i): *The map π_H is ess. category preserving.*
- (ii): *$\pi_H(F')$ is a typical set.*

Proof. The implication (i) \Rightarrow (ii) is obvious. To get (ii) \Rightarrow (i), we first argue as in [27], Proposition A.7. Namely, F' is a Polish space endowed with the induced topology. Fix a countable dense set $\Omega' \subseteq \Omega$, and an open non-empty subset O of F' . Then $\pi_H(F') = \pi_H(\Omega'^{-1} \cdot O) = \Omega'^{-1} \cdot \pi_H(O)$. It implies that $\pi_H(O')$ is not meager for every non-empty open subset O' of Ω_G .

Assume the map π_H is not ess. category preserving. Then $\pi_H(B)$ is meager for some set $B \in \Omega_G$ of the second category. Take a collection C_i , $i \in \mathbb{N}$, of closed nowhere-dense sets such that $\pi_H(B) \subseteq \cup_i C_i$. Then a closed set $\pi_H^{-1}(C_{i_0})$ is not meager for some i_0 . It implies that there is a non-empty open set $O^* \subseteq \pi_H^{-1}(C_{i_0})$. We have a contradiction and Proposition 7.1 is proved. \square

How much can we extend a typical H -action? Obviously, π_H is still ess. category preserving if we replace G with any free product, say $G * G_1$, since the topological Fubini theorem applies. Besides, ess. category preserving maps π_H^G (i.e. π_H defined on Ω_G) are closed under taking inductive limits in the following sense.

Proposition 7.2. *Let $H \leq G_1 \leq \dots \leq G_n \leq \dots$ be a countable chain of countable groups $H, G_i, i \in \mathbb{N}$. Suppose all $\pi_H^{G_i}$ are ess. category preserving; then $\pi_H^{\lim_{i \rightarrow \infty} G_i}$ is ess. category preserving as well.*

Proof. Denote

$$B = \bigcap_i \pi_H^{G_i}(F_{G_i}), \quad B_i = F_{G_i} \cap \bigcap_{j>i} \pi_{G_i}^{G_j}(F_{G_j}).$$

Every H -action in B can be extended to some element of B_1 , every G_1 -action in B_1 can be extended to some element of B_2 , and so on. It means that we obtained some subset of $\Omega_{\lim_{i \rightarrow \infty} G_i}$, say C , satisfying $\pi_H(C) = B$. The reader will easily prove that $C \subseteq F'_{\lim_{i \rightarrow \infty} G_i}$. To conclude the proof, it remains to apply Proposition 7.1. \square

Proposition 7.3. *Let G be any countable non-abelian group, H its normal abelian subgroup containing an element h_0 of infinite order. Then π_H^G is not ess. category preserving.*

Proof. Assume H is included in the center of G . Since, for a typical H -action T , the centralizer $C(T)$ is abelian, any extension of T to a G -action must be abelian. Therefore its orbit under conjugacies is not dense in Ω_G . So, $\pi_H^G(F')$ is meager.

If H is not in the center, then we can choose $h_1, h_2 \in H$, $g \in G$ such that $g^{-1}h_1g = h_2$, $h_1 \neq h_2$. If, in addition, $g^{-1}h_0^k g = h_0^k$ for some $k \neq 0$, then a typical H -action T can not be extended to any G -action. Indeed, otherwise take some such extension T , then $T_{h_i}, T_g \in CL\{T_{kh_0}^n : n \in \mathbb{Z}\}$ because of Theorem 6.2. Hence, $T_{h_2} = T_g^{-1}T_{h_1}T_g = T_{h_1}$, and we have a contradiction with the freeness of a typical H -action.

Assume $g^{-1}h_0g = h_3$, $h_0^k \neq h_3^k$ for any $k \neq 0$. Pick a positive k such that $H' = \langle h_0^k, h_3^k \rangle$ is a free abelian subgroup. It is well known that for a typical

transformation all its powers are not isomorphic to each other (see, for example [18]). It follows that a typical H' -action does not admit any isomorphism between $T_{h_0^k}$ and $T_{h_3^k}$ if H' is cyclic. The case H' has two generators is analogous. By Theorem 6.2, we conclude that there is no isomorphism between $T_{h_0^k}$ and $T_{h_3^k}$ for a typical H -action T . Hence T can not be extended to any G -action. This completes the proof. \square

It is natural to look at possible analogs of Theorems 6.2 and 6.7 if we omit the restriction to be abelian for H and G . Let us list a series of examples somewhat indicating why there is currently no complete answer even for $H = \mathbb{Z}$.

1. The most closed to abelian groups are finite extensions of them. Obviously, for every positive integer n

$$G_n = \text{gr}\langle t_1, \dots, t_n, s; (\forall i, j)[t_i t_j = t_j t_i \& s t_i s^{-1} = t_i^{-1}] \rangle$$

contains a free abelian subgroup $H = \langle t_1, \dots, t_n, s^2 \rangle$ of index 2. Arguing as in Remark 6.4, we see that $\pi_{H'}^{G_n}(\Omega_{G_n})$ is meager for every infinite subgroup $H' \leq H$.

Besides, it was proved in [4] that for a typical action T of

$$G_n = \text{gr}\langle t, s; (\forall i, j)[t_i, t_j] = 1 = t_0 \dots t_{n-1}] \rangle,$$

where $t_i = s^i t s^{-i}$, $n > 1$, the transformation T_{s^n} has homogeneous spectrum of multiplicity n (see slightly modified versions of G_n with the same property in [11], [29]). The groups G_n are also finite extensions of abelian. Since a typical transformation has simple spectrum, $\pi_{\langle s^k \rangle}^{G_n}$ is not ess. category preserving for every $k \neq 0$. On the other hand, every transformation T_s can be extended to a G_n -action if we put the identity as T_t . So, $\pi_{\langle s^k \rangle}^{G_n}(\Omega_{G_n})$ is typical for every $k \neq 0$.

2. Our technique, working well for abelian groups, was also based on approximations by finite actions. Since there exist countable groups with no non-trivial irreducible finitely dimensional unitary representations, it may happen that some non-trivial countable groups do not admit finite actions except an action by identities. However all the above can (in)directly work for some groups that are not even virtually abelian. Take, for example,

$$G = \text{gr}\langle t_1, s; t_1 s = s t_1^2 \rangle.$$

It is well known (see, for example [5], [18]) that a typical transformation is spectrally disjoint to its square. Thus $\pi_{\langle t_1 \rangle}^G(\Omega_G)$ is meager. Besides, take

$$G^* = \text{gr}\langle t_1, s, t_2; t_i s = s t_i^2, i = 1, 2 \& t_1 t_2 = t_2 t_1 \rangle.$$

We claim that $\pi_G^{G^*}$ is ess. category preserving. Indeed, first note that G^* being solvable is an amenable group. Observe that, for a typical G -action T , T_{t_1} has rank 1 and is *rigid* (i.e. $T_{t_1}^{k_j}$ tends to the identity for some sequence k_j) (see [5]). It follows that the centralizer

$$C\{T_{t_1}\} = CL\{T_{nt_1} : n \in \mathbb{Z}\}$$

is uncountable and has no isolated points. Moreover, by the standard topological arguments, it can be then shown that the monothetic group $C\{T_{t_1}\}$ can not be covered by the countable union of closed sets $T_{j t_1} B_i$, $j \in \mathbb{Z}, i \in \mathbb{N}$, where B_i consists of all elements S of the centralizer satisfying S^i is the identity. Pick an element, say S , which is not covered. Then $\langle T_{t_1}, S \rangle$ form a free \mathbb{Z}^2 -action. It easily implies that we extended T to a free G^* -action by putting $T_{t_2} = S$. Applying Proposition 7.1, we conclude that $\pi_G^{G^*}$ is ess. category preserving.

3. Take a free product of cyclic groups of order 2 and 3, i.e.

$$G = \text{gr}\langle t, s; t^2, s^3 \rangle.$$

Every transformation is a composition two periodic transformations, say S_2 and S_3 . Moreover, with no loss of the generality, S_i can be chosen satisfying S_i^i ($i = 2, 3$) are the identity (see [29]). It implies that $\pi_{<ts>}^G(\Omega_G)$ is typical.

4. Recall that for an inclusion $H \leq G$ of countable, discrete groups one says that the pair (G, H) has **relative property (T)** if there exists $\delta > 0$ and a finite $B \subset G$ such that if π is a unitary representation of G in a Hilbert space and f is a unit vector satisfying

$$(\forall g \in B) \|\pi(g)(f) - f\| < \delta,$$

then there exists a vector f_0 such that $\pi(g)(f_0) = f_0$ for any $g \in H$. One says that G has **property (T)** and is called Kazhdan, if the pair (G, G) has relative property (T).

Take the pair $H \leq G$ with relative property (T), where H is not Kazhdan. For a typical G -action T , $T|_H$ is not ergodic, and a typical H -action is ergodic (see [20]). It implies that $\pi_H(F')$ is meager.

One may ask another natural question about how many extensions for a typical H -action we could expect. Arguing in a more or less standard way, it can be easily answered for countable abelian groups G (and H). Let us only stress that there appear new effects (i.e. not the continuum or nothing). For example, let $G = H \oplus \mathbb{Z}/3\mathbb{Z}$, $H = \oplus^\infty \mathbb{Z}/2\mathbb{Z}$. By Theorem 4.5, for a typical H -action T the centralizer of T consists of elements of order 2. Therefore T has the only trivial extension to a G -action.

It would be interesting to see any differences in the classifications of pairs $(H \leq G)$ according to Theorems 6.2, 6.7, 6.8 if we take representations of groups by unitary operators of the separable infinite-dimensional Hilbert space or by isometries of the universal Urysohn metric space. Besides, it is expected that the classification of pairs will be essentially different for actions by measure preserving homeomorphisms of the Cantor set.

It would be also interesting to see if there exists one more characteristic dynamic property for π_H in the above sense.

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